KAM normal form I

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Disclaimer: everything will "live" in $\mathbb{R}^n$ with "periodicity conditions" but non-trivial dynamics occur on $\mathbb{T}^n$ and some properties are easier to formulate on $\mathbb{T}^n$ ($\mathbb{T}^n$ is compact with no boundary).

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The vector $\alpha \in \mathbb{T}^n$ is non-resonant iff the translation $T_\alpha : \mathbb{T}^n \to \mathbb{T}^n$ is uniquely ergodic: it has a unique invariant Borel probability measure (which is the Lebesgue or Haar measure m).

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For $k \in \mathbb{Z}^n$, define the Fourier coefficient $f_k \in \mathbb{C}$ by

$$f_k := \int_{\mathbb{T}^n} e^{-2\pi i k\cdot x} f(x)dx, \quad f_0 = [f] = \int_{\mathbb{T}^n} f(x)dx, \quad \int_{\mathbb{T}^n} := \int_{[0,1]^n}$$

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The invariance of f gives $e^{2\pi i k \cdot \alpha} f_k = f_k$: for $k \neq 0$, $k \cdot \alpha \notin \mathbb{Z}$ by the non-resonance condition so $e^{2\pi i k \cdot \alpha} \neq 1$ so $f_k = 0$.

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If α is resonant, find a non-constant invariant continuous function and thus an invariant measure different from m.

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For $\gamma > 0$ and $\tau > 0$, let K_γ^τ be the set of $\alpha \in \mathbb{T}^n$ such that for all $k \in \mathbb{Z}^n \setminus \{0\}$

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Lemma

For $\tau > n$, K^τ has full Lebesgue measure. More precisely, there exists $C = C(\tau) > 1$ such that

$$m(\mathbb{T}^n \setminus K_\gamma^\tau) \leq C\gamma.$$

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and therefore

$$m(\mathbb{T}^n \setminus K_\gamma^\tau) \leq \sum_{k \in \mathbb{Z}^n \setminus \{0\}} m(B_k) \leq 2\gamma \sum_{k \in \mathbb{Z}^n \setminus \{0\}} \frac{1}{|k|^\tau} = C\gamma$$

where $C < +\infty$ since $\tau > n$.

Diffeomorphism of the torus

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Lemma

Every continuous map $F : \mathbb{T}^n \to \mathbb{T}^n$ *has a lift of the form*

$$F(x) = Ax + P(x), \quad A \in M_n(\mathbb{Z}), \quad P : \mathbb{T}^n \to \mathbb{R}^n$$

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Let $\Phi_1, \Phi_2 : \mathbb{T}^n \to \mathbb{T}^n$ *homeomorphisms homotopic to the identity*

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What is true is that if α is Diophantine and F is analytic (or sufficiently smooth), then there exists $U : \mathbb{T}^n \to \mathbb{R}^n$ and $v \in \mathbb{R}^n$ close to zero such that $\Phi = \mathrm{Id} + U$ conjugates $F - v = T_{\alpha-v} + P$ to T_α.

KAM normal form

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Theorem (Arnold)

Fix $\gamma > 0$ and $\tau > n$. Then for any $\alpha \in K_\gamma^\tau$ and any real-analytic $P : \mathbb{T}^n \to \mathbb{R}^n$ sufficiently close to zero, there exist a unique couple (U, v) close to zero, where $U : \mathbb{T}^n \to \mathbb{R}^n$ is real-analytic with zero average and $v \in \mathbb{R}^n$, such that for $F = T_\alpha + P$ and $\Phi = \mathrm{Id} + U$, we have

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Beware that U and v are uniquely defined, and that the conjugacy holds true, only for $\alpha \in K_\gamma^\tau$!

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(1) If we further assume that F has a rotation vector equals to α, then $v = 0$ and F is analytically conjugated to T_α
(2) If we include F into any "non-degenerate" family $(F_I)_{I \in \mathbb{T}^n}$ then the set of $I \in \mathbb{T}^n$ for which F_I is analytically conjugated to a Diophantine translation has positive Lebesgue measure