KAM normal form II

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Theorem (Arnold)

Fix $\gamma > 0$ and $\tau > n$. Then for any $\alpha \in K_\gamma^\tau$ and any real-analytic $P : \mathbb{T}^n \to \mathbb{R}^n$ sufficiently close to zero, there exist a unique couple (U, v) close to zero, where $U : \mathbb{T}^n \to \mathbb{R}^n$ is real-analytic with zero average and $v \in \mathbb{R}^n$, such that for $F = T_\alpha + P$ and $\Phi = \mathrm{Id} + U$, we have

$$\Phi \circ (F - v) \circ \Phi^{-1} = T_\alpha.$$

Moreover, $U = U(\alpha, P)$ and $v = v(\alpha, P)$ depend
(1) smoothly on α (extend to smooth functions in $\alpha \in \mathbb{T}^n$)
(2) analytically in P (analytic in $\varepsilon \in \mathbb{R}$ if $\varepsilon \mapsto P_\varepsilon$ is analytic)
(3) and we have the estimates

$$\mathrm{Lip}_\theta(U) \leq 1/2, \quad \mathrm{Lip}_\alpha(u) \leq 1/2.$$

Beware that U and v are uniquely defined, and that the conjugacy holds true, only for $\alpha \in K_\gamma^\tau$!

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$$\mu(F) := \int_{\mathbb{T}^n} (F - \mathrm{Id}) d\mu \mod \mathbb{Z}^n, \quad \mathrm{Rot}(F) := \{\mu(F) \in \mathbb{T}^n \mid \mu \in \mathcal{M}(F)\}.$$

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Corollary 1

Under the assumptions of the theorem, assume further that $\alpha \in \mathrm{Rot}(F)$. Then $v = v(F) = 0$, that is F is analytically conjugated to T_α.

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Let $F_* = \Phi \circ F \circ \Phi^{-1}$, then

$$F_* = \mathrm{Id} + \alpha + v + X, \quad X := U \circ (F \circ \Phi^{-1}) - U \circ (F \circ \Phi^{-1} - v).$$

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Taking the supremum norm on $\mathbb{T}^n$ this gives

$$|v| \leq \sup_{\theta \in \mathbb{T}^n} |X(\theta)| \leq \mathrm{Lip}(U)|v| \leq |v|/2.$$

Hence $v = 0$.

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Fix $\alpha_0 \in K_\gamma^\tau$, $P : \mathbb{T}^n \to \mathbb{R}^n$ real-analytic and consider the following "deformation" of $F_\varepsilon = T_{\alpha_0} + \varepsilon P$, $\varepsilon \in \mathbb{R}$:

$$\theta \in \mathbb{T}^n \longmapsto F_\varepsilon(\theta) + I = \theta + \alpha_0 + I + \varepsilon P(\theta) \in \mathbb{T}^n$$

where $I \in \mathbb{T}^n$ are parameters.

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Corollary 2 (Arnold)

There exists $\varepsilon_0 = \varepsilon_0(\gamma, \tau, P) > 0$ and $\tilde{C} = \tilde{C}(\tau) > 1$ such that for all $|\varepsilon| \leq \varepsilon_0$, if $\tilde{K}_{\gamma,\varepsilon}^\tau$ is the set of $I \in \mathbb{T}^n$ such that $A_{I,\varepsilon}$ is analytically conjugated to T_α for some $\alpha \in K_\gamma^\tau$, then

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We will see that $\varepsilon_0 = c(\tau, P)\gamma = c\gamma$ for some $c < 1$, so choosing $\gamma = \varepsilon/c$ we have $m\left(\mathbb{T}^n \setminus \tilde{K}_{\varepsilon/c,\varepsilon}^\tau\right) \leq \tilde{C}\varepsilon/c$ for ε small enough.

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Corollary 2 holds true if we replace $I \mapsto I$ in the definition of $A_{I,\varepsilon}$ by a local C^1-diffeomorphism $I \in \mathbb{R}^n \mapsto \omega(I) \in \mathbb{R}^n$ ("unperturbed frequency map"), provided $\varepsilon_0 = \varepsilon_0(\gamma, \tau, P, \omega)$.

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Also true under weaker non-degeneracy: $I \in \mathbb{R}^d$ for some $d \geq 1$ and $\omega : \mathbb{R}^d \to \mathbb{R}^n$ is "curved": all partial derivatives of ω of order ≥ 1 generate $\mathbb{R}^n$ (Rüssmann). This requires smoothness of $\alpha \mapsto v(\alpha, \varepsilon)$.

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Let P_ε an analytic family in $\varepsilon \in \mathbb{R}^n$ with $P_0 = 0$ and consider

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$$\mathrm{Lip}(v) = \sup_{I \in \mathbb{T}^n} |Dv(I)| := \sup_{I \in \mathbb{T}^n} \sup_{e \in \mathbb{R}^n,\ |e|=1} |Dv(I)e| = \sup_{I \in \mathbb{T}^n} \sum_{|l|=1} |\partial^l v(I)|.$$

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$\mathrm{Lip}(v_1 + v_2) \leq \mathrm{Lip}(v_1) + \mathrm{Lip}(v_2)$
$\mathrm{Lip}(v_1 \circ (\mathrm{Id} + v_2)) \leq \mathrm{Lip}(v_1)(1 + \mathrm{Lip}(v_2))$.

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Let (E, d) be a metric space, $K \subseteq E$ a subset and $v : K \to \mathbb{R}^n$ Lipschitz. Then v extends to $\hat{v} : E \to \mathbb{R}^n$ with $\mathrm{Lip}(\hat{v}) = \mathrm{Lip}(v)$.

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For simplicity, we shall write $\hat{v} = v$. The extension is not unique!

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$\mathcal{P}_v$ contracts B^*: $|\mathcal{P}_v(u_1) - \mathcal{P}_v(u_2)|_{C^0} \leq \mathrm{Lip}(v)|u_1 - u_2|_{C^0}$. □

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Recall the statement we want to prove.

Corollary 2 (Arnold)

There exists $\varepsilon_0 = \varepsilon_0(\gamma, \tau, P) > 0$ and $\tilde{C} = \tilde{C}(\tau) > 1$ such that for all $|\varepsilon| \leq \varepsilon_0$, if $\tilde{K}^{\tau}_{\gamma,\varepsilon}$ is the set of $I \in \mathbb{T}^n$ such that $A_{I,\varepsilon}$ is analytically conjugated to T_α for some $\alpha \in K^{\tau}_{\gamma}$, then

$$m\left(\mathbb{T}^n \setminus \tilde{K}^{\tau}_{\gamma,\varepsilon}\right) \leq \tilde{C}\gamma.$$

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so applying Lemma 3 to $S = \mathbb{T}^n \setminus K$ we arrive at

$$m(\mathbb{T}^n \setminus \tilde{K}_\varepsilon) = m(\psi_\varepsilon(\mathbb{T}^n \setminus K)) \leq (3/2)^n m(\mathbb{T}^n \setminus K) \leq \tilde{C}\gamma.$$

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