KAM normal form III

Abed Bounemoura

CNRS-UPPA

September 19, 2025

Analytic functions

Analytic functions

A smooth vector field $P : \mathbb{T}^n \to \mathbb{R}^n$ equals its Fourier series:

$$P(\theta) = \sum_{k \in \mathbb{Z}^n} P_k e^{2\pi i k \cdot \theta}, \quad P_k := \int_{\mathbb{T}^n} P(\theta) e^{-2\pi i k \cdot \theta} dx = \overline{P_{-k}} \in \mathbb{C}^n.$$

Analytic functions

A smooth vector field $P : \mathbb{T}^n \to \mathbb{R}^n$ equals its Fourier series:

$$P(\theta) = \sum_{k \in \mathbb{Z}^n} P_k e^{2\pi i k \cdot \theta}, \quad P_k := \int_{\mathbb{T}^n} P(\theta) e^{-2\pi i k \cdot \theta} dx = \overline{P_{-k}} \in \mathbb{C}^n.$$

We say that P is analytic if there exists $s > 0$ such that

$$|P|_s := \sum_{k \in \mathbb{Z}^n} |P_k| e^{2\pi s |k|} < +\infty.$$

Analytic functions

A smooth vector field $P : \mathbb{T}^n \to \mathbb{R}^n$ equals its Fourier series:

$$P(\theta) = \sum_{k \in \mathbb{Z}^n} P_k e^{2\pi i k \cdot \theta}, \quad P_k := \int_{\mathbb{T}^n} P(\theta) e^{-2\pi i k \cdot \theta} dx = \overline{P_{-k}} \in \mathbb{C}^n.$$

We say that P is analytic if there exists $s > 0$ such that

$$|P|_s := \sum_{k \in \mathbb{Z}^n} |P_k| e^{2\pi s |k|} < +\infty.$$

Convention: for $k = (k_1, \dots, k_n) \in \mathbb{Z}^n$, $x = (x_1, \dots, x_n) \in \mathbb{C}^n$

$$|k| = |k_1| + \cdots |k_n|, \quad |x| = \sup\{|x_1|, \dots, |x_n|\}.$$

Analytic functions

A smooth vector field $P : \mathbb{T}^n \to \mathbb{R}^n$ equals its Fourier series:

$$P(\theta) = \sum_{k \in \mathbb{Z}^n} P_k e^{2\pi i k \cdot \theta}, \quad P_k := \int_{\mathbb{T}^n} P(\theta) e^{-2\pi i k \cdot \theta} dx = \overline{P_{-k}} \in \mathbb{C}^n.$$

We say that P is analytic if there exists $s > 0$ such that

$$|P|_s := \sum_{k \in \mathbb{Z}^n} |P_k| e^{2\pi s |k|} < +\infty.$$

Convention: for $k = (k_1, \ldots, k_n) \in \mathbb{Z}^n$, $x = (x_1, \ldots, x_n) \in \mathbb{C}^n$

$$|k| = |k_1| + \cdots |k_n|, \quad |x| = \sup\{|x_1|, \ldots, |x_n|\}.$$

If we define complex s-neighborhood of $\mathbb{R}^n$ and $\mathbb{T}^n$ by

$$\mathbb{R}^n_s := \{x \in \mathbb{C}^n \mid d(x, \mathbb{R}^n) = |\mathrm{Im}\, x| < s\}, \quad \mathbb{T}^n_s := \mathbb{R}^n_s / \mathbb{Z}^n$$

Analytic functions

A smooth vector field $P : \mathbb{T}^n \to \mathbb{R}^n$ equals its Fourier series:

$$P(\theta) = \sum_{k \in \mathbb{Z}^n} P_k e^{2\pi i k \cdot \theta}, \quad P_k := \int_{\mathbb{T}^n} P(\theta) e^{-2\pi i k \cdot \theta} dx = \overline{P_{-k}} \in \mathbb{C}^n.$$

We say that P is analytic if there exists $s > 0$ such that

$$|P|_s := \sum_{k \in \mathbb{Z}^n} |P_k| e^{2\pi s |k|} < +\infty.$$

Convention: for $k = (k_1, \dots, k_n) \in \mathbb{Z}^n$, $x = (x_1, \dots, x_n) \in \mathbb{C}^n$

$$|k| = |k_1| + \cdots |k_n|, \quad |x| = \sup\{|x_1|, \dots, |x_n|\}.$$

If we define complex s-neighborhood of $\mathbb{R}^n$ and $\mathbb{T}^n$ by

$$\mathbb{R}^n_s := \{x \in \mathbb{C}^n \mid d(x, \mathbb{R}^n) = |\mathrm{Im}\, x| < s\}, \quad \mathbb{T}^n_s := \mathbb{R}^n_s / \mathbb{Z}^n$$

then P extends to a holomorphic vector field $P : \mathbb{T}^n_s \to \mathbb{C}^n$ and

$$||P||_s := \sup_{\theta \in \mathbb{T}^n_s} |P(\theta)| \leq |P|_s.$$

KAM normal form

KAM normal form

Theorem (Arnold)

Fix $\gamma > 0$, $\tau > n$ *and* $0 < s \leq 1$, *and define*

$$\bar{C} := (\tau+1)!4^{-1}(2\pi)^{-\tau}, \quad C_* := \bar{C} n 2^6 16^{\tau+1} > 1, \quad c_* = (4C_*)^{-1} < 1.$$

KAM normal form

Theorem (Arnold)

Fix $\gamma > 0$, $\tau > n$ and $0 < s \leq 1$, and define

$$\bar{C} := (\tau+1)!4^{-1}(2\pi)^{-\tau}, \quad C_* := \bar{C}n2^6 16^{\tau+1} > 1, \quad c_* = (4C_*)^{-1} < 1.$$

Then for any $\alpha \in K_\gamma^\tau$ and any real-analytic $P : \mathbb{T}^n \to \mathbb{R}^n$ satisfying

$$\varepsilon := |P|_s \leq c_* \gamma s^{\tau+1}$$

there exist a unique couple (U, v), where $U : \mathbb{T}^n \to \mathbb{R}^n$ is real-analytic with zero average and $v \in \mathbb{R}^n$, such that for $F = T_\alpha + P$, $\Phi = \mathrm{Id} + U$ and $\Psi = \Phi^{-1} = \mathrm{Id} - V$, we have

$$\Phi \circ (F - v) \circ \Phi^{-1} = T_\alpha.$$

KAM normal form

Theorem (Arnold)

Fix $\gamma > 0$, $\tau > n$ *and* $0 < s \leq 1$, *and define*

$$\bar{C} := (\tau+1)! 4^{-1}(2\pi)^{-\tau}, \quad C_* := \bar{C} n 2^6 16^{\tau+1} > 1, \quad c_* = (4C_*)^{-1} < 1.$$

Then for any $\alpha \in K_\gamma^\tau$ *and any real-analytic* $P : \mathbb{T}^n \to \mathbb{R}^n$ *satisfying*

$$\varepsilon := |P|_s \leq c_* \gamma s^{\tau+1}$$

there exist a unique couple (U, v), *where* $U : \mathbb{T}^n \to \mathbb{R}^n$ *is real-analytic with zero average and* $v \in \mathbb{R}^n$, *such that for* $F = T_\alpha + P$, $\Phi = \mathrm{Id} + U$ *and* $\Psi = \Phi^{-1} = \mathrm{Id} - V$, *we have*

$$\Phi \circ (F - v) \circ \Phi^{-1} = T_\alpha.$$

with the estimates

$$|V|_{s/2} \leq |U|_{s/2} \leq \frac{C_* \varepsilon}{\gamma s^\tau} \leq s/4, \quad |DV|_{s/2} \leq 2|DU|_{s/2} \leq \frac{2C_* \varepsilon}{\gamma s^{\tau+1}} \leq 1/2,$$

$$|v| \leq 2\varepsilon, \quad \mathrm{Lip}_\alpha(v) \leq \frac{2C_*\varepsilon}{\gamma s^{\tau+1}} \leq 1/2.$$

Properties of analytic functions (Fourier norm)

Properties of analytic functions (Fourier norm)

(1) Product. $P, Q : \mathbb{T}^n_s \to \mathbb{C}$, $|PQ|_s \leq |P|_s |Q|_s$

Properties of analytic functions (Fourier norm)

(1) Product. $P, Q : \mathbb{T}^n_s \to \mathbb{C}$, $|PQ|_s \leq |P|_s |Q|_s$

(2) Derivative. $P : \mathbb{T}^n_s \to \mathbb{C}^n$ and $0 < \sigma \leq s$

$$\sum_{l \in \mathbb{N}^n} \frac{\sigma^{|l|}}{l!} |\partial^l P|_{s-\sigma} \leq |P|_s, \quad |DP|_{s-\sigma} := \sum_{|l|=1} |\partial^l P|_{s-\sigma} \leq \sigma^{-1} |P|_s.$$

Properties of analytic functions (Fourier norm)

(1) Product. $P, Q : \mathbb{T}^n_s \to \mathbb{C}$, $|PQ|_s \leq |P|_s |Q|_s$

(2) Derivative. $P : \mathbb{T}^n_s \to \mathbb{C}^n$ and $0 < \sigma \leq s$

$$\sum_{l \in \mathbb{N}^n} \frac{\sigma^{|l|}}{l!} |\partial^l P|_{s-\sigma} \leq |P|_s, \quad |DP|_{s-\sigma} := \sum_{|l|=1} |\partial^l P|_{s-\sigma} \leq \sigma^{-1} |P|_s.$$

(3) Composition. $0 < \sigma \leq s$, $U : \mathbb{T}^n_{s-\sigma} \to \mathbb{C}^n$, $|U|_{s-\sigma} \leq \sigma$

$$|P \circ (\mathrm{Id} + U)|_{s-\sigma} \leq |P|_s, \quad |P \circ (\mathrm{Id} + U) - P|_{s-\sigma} \leq \sigma^{-1} |U|_{s-\sigma} |P|_s.$$

Properties of analytic functions (Fourier norm)

(1) Product. $P, Q : \mathbb{T}^n_s \to \mathbb{C}$, $|PQ|_s \leq |P|_s |Q|_s$

(2) Derivative. $P : \mathbb{T}^n_s \to \mathbb{C}^n$ and $0 < \sigma \leq s$

$$\sum_{l \in \mathbb{N}^n} \frac{\sigma^{|l|}}{l!} |\partial^l P|_{s-\sigma} \leq |P|_s, \quad |DP|_{s-\sigma} := \sum_{|l|=1} |\partial^l P|_{s-\sigma} \leq \sigma^{-1} |P|_s.$$

(3) Composition. $0 < \sigma \leq s$, $U : \mathbb{T}^n_{s-\sigma} \to \mathbb{C}^n$, $|U|_{s-\sigma} \leq \sigma$

$$|P \circ (\mathrm{Id} + U)|_{s-\sigma} \leq |P|_s, \quad |P \circ (\mathrm{Id} + U) - P|_{s-\sigma} \leq \sigma^{-1} |U|_{s-\sigma} |P|_s.$$

(4) Taylor. $0 < \sigma \leq s$, $U_1, U_2 : \mathbb{T}^n_{s-\sigma} \to \mathbb{C}^n$, $|U_i|_{s-\sigma} \leq \sigma$

$$|P \circ (\mathrm{Id} + U_1) - P \circ (\mathrm{Id} + U_2)|_{s-\sigma} \leq |DP|_s |U_1 - U_2|_{s-\sigma}.$$

Properties of analytic functions (Fourier norm)

(1) Product. $P, Q : \mathbb{T}^n_s \to \mathbb{C}$, $|PQ|_s \le |P|_s |Q|_s$

(2) Derivative. $P : \mathbb{T}^n_s \to \mathbb{C}^n$ and $0 < \sigma \le s$

$$\sum_{l \in \mathbb{N}^n} \frac{\sigma^{|l|}}{l!} |\partial^l P|_{s-\sigma} \le |P|_s, \quad |DP|_{s-\sigma} := \sum_{|l|=1} |\partial^l P|_{s-\sigma} \le \sigma^{-1} |P|_s.$$

(3) Composition. $0 < \sigma \le s$, $U : \mathbb{T}^n_{s-\sigma} \to \mathbb{C}^n$, $|U|_{s-\sigma} \le \sigma$

$$|P \circ (\mathrm{Id} + U)|_{s-\sigma} \le |P|_s, \quad |P \circ (\mathrm{Id} + U) - P|_{s-\sigma} \le \sigma^{-1} |U|_{s-\sigma} |P|_s.$$

(4) Taylor. $0 < \sigma \le s$, $U_1, U_2 : \mathbb{T}^n_{s-\sigma} \to \mathbb{C}^n$, $|U_i|_{s-\sigma} \le \sigma$

$$|P \circ (\mathrm{Id} + U_1) - P \circ (\mathrm{Id} + U_2)|_{s-\sigma} \le |DP|_s |U_1 - U_2|_{s-\sigma}.$$

(5) Inverse. $0 < 2\sigma \le s$, $|U|_{s-\sigma} \le \sigma$, $|DU|_{s-\sigma} < 1$, then $\Phi = \mathrm{Id} + U : \mathbb{T}^n_{s-\sigma} \to \mathbb{C}^n$ is an analytic embedding such that $\mathbb{T}^n_{s-2\sigma} \subseteq \Phi(\mathbb{T}^n_{s-\sigma}) \subseteq \mathbb{T}^n_s$ and $\Phi^{-1} = \mathrm{Id} - V$ with

$$|V|_{s-2\sigma} \le |U|_{s-\sigma}, \quad |DV|_{s-2\sigma} \le |DU|_{s-\sigma} (1 - |DU|_{s-\sigma})^{-1}.$$

Properties of analytic functions: proof ($n = 1$)

Properties of analytic functions: proof ($n = 1$)

Points (1) and (2):

Properties of analytic functions: proof ($n = 1$)

Points (1) and (2):

$$|PQ|_s = \sum_{k\in\mathbb{Z}} \left| \sum_{m\leq k} P_m Q_{k-m} \right| e^{2\pi s|k|} \leq \sum_{k,m} |P_m||Q_{k-m}||e^{2\pi s|m|}|e^{2\pi s|k-m|}$$

Properties of analytic functions: proof ($n = 1$)

Points (1) and (2):

$$|PQ|_s = \sum_{k\in\mathbb{Z}} \left| \sum_{m\leq k} P_m Q_{k-m} \right| e^{2\pi s|k|} \leq \sum_{k,m} |P_m||Q_{k-m}||e^{2\pi s|m|}|e^{2\pi s|k-m|}$$

$$\sum_{l\in\mathbb{N}} (l!)^{-1}\sigma^l |\partial^l P|_{s-\sigma} \leq \sum_{l\in\mathbb{N}} \sum_{k\in\mathbb{Z}} (l!)^{-1}(\sigma 2\pi|k|)^l |P_k| e^{2\pi(s-\sigma)|k|} = |P|_s.$$

Properties of analytic functions: proof ($n = 1$)

Points (1) and (2):

$$|PQ|_s = \sum_{k \in \mathbb{Z}} \left| \sum_{m \leq k} P_m Q_{k-m} \right| e^{2\pi s|k|} \leq \sum_{k,m} |P_m||Q_{k-m}||e^{2\pi s|m|}|e^{2\pi s|k-m|}$$

$$\sum_{l \in \mathbb{N}} (l!)^{-1} \sigma^l |\partial^l P|_{s-\sigma} \leq \sum_{l \in \mathbb{N}} \sum_{k \in \mathbb{Z}} (l!)^{-1} (\sigma 2\pi |k|)^l |P_k| e^{2\pi(s-\sigma)|k|} = |P|_s.$$

Point (3) follows from (1), (2) and analyticity: we can write

$$P(\theta + X(\theta)) = \sum_{l \in \mathbb{N}} (l!)^{-1} (\partial^l P(\theta)) X^l(\theta), \quad \theta \in \mathbb{T}^n_{s-\sigma}$$

Properties of analytic functions: proof ($n = 1$)

Points (1) and (2):

$$|PQ|_s = \sum_{k\in\mathbb{Z}} \left| \sum_{m\leq k} P_m Q_{k-m} \right| e^{2\pi s|k|} \leq \sum_{k,m} |P_m||Q_{k-m}||e^{2\pi s|m|}|e^{2\pi s|k-m|}$$

$$\sum_{l\in\mathbb{N}} (l!)^{-1}\sigma^l |\partial^l P|_{s-\sigma} \leq \sum_{l\in\mathbb{N}} \sum_{k\in\mathbb{Z}} (l!)^{-1}(\sigma 2\pi|k|)^l |P_k| e^{2\pi(s-\sigma)|k|} = |P|_s.$$

Point (3) follows from (1), (2) and analyticity: we can write

$$P(\theta + X(\theta)) = \sum_{l\in\mathbb{N}} (l!)^{-1}(\partial^l P(\theta))X^l(\theta), \quad \theta \in \mathbb{T}^n_{s-\sigma}$$

$$|P \circ (\mathrm{Id} + X)|_{s-\sigma} \leq \sum_{l\in\mathbb{N}} (l!)^{-1} |\partial^l P|_{s-\sigma} |X|^l_{s-\sigma} \leq |P|_s$$

Properties of analytic functions: proof ($n = 1$)

Points (1) and (2):

$$|PQ|_s = \sum_{k \in \mathbb{Z}} \left| \sum_{m \leq k} P_m Q_{k-m} \right| e^{2\pi s|k|} \leq \sum_{k,m} |P_m||Q_{k-m}||e^{2\pi s|m|}|e^{2\pi s|k-m|}$$

$$\sum_{l \in \mathbb{N}} (l!)^{-1} \sigma^l |\partial^l P|_{s-\sigma} \leq \sum_{l \in \mathbb{N}} \sum_{k \in \mathbb{Z}} (l!)^{-1} (\sigma 2\pi |k|)^l |P_k| e^{2\pi(s-\sigma)|k|} = |P|_s.$$

Point (3) follows from (1), (2) and analyticity: we can write

$$P(\theta + X(\theta)) = \sum_{l \in \mathbb{N}} (l!)^{-1} (\partial^l P(\theta)) X^l(\theta), \quad \theta \in \mathbb{T}^n_{s-\sigma}$$

$$|P \circ (\mathrm{Id} + X)|_{s-\sigma} \leq \sum_{l \in \mathbb{N}} (l!)^{-1} |\partial^l P|_{s-\sigma} |X|^l_{s-\sigma} \leq |P|_s$$

$$|P \circ (\mathrm{Id} + X) - P|_{s-\sigma} \leq \sum_{l \geq 1} (l!)^{-1} |\partial^l P|_{s-\sigma} ||X|^l_{s-\sigma} \leq \sigma^{-1} |X|_{s-\sigma} |P|_s.$$

Properties of analytic functions: proof ($n = 1$)

Points (1) and (2):

$$|PQ|_s = \sum_{k\in\mathbb{Z}} \left| \sum_{m\leq k} P_m Q_{k-m} \right| e^{2\pi s|k|} \leq \sum_{k,m} |P_m||Q_{k-m}||e^{2\pi s|m|}|e^{2\pi s|k-m|}$$

$$\sum_{l\in\mathbb{N}} (l!)^{-1}\sigma^l |\partial^l P|_{s-\sigma} \leq \sum_{l\in\mathbb{N}} \sum_{k\in\mathbb{Z}} (l!)^{-1}(\sigma 2\pi|k|)^l |P_k| e^{2\pi(s-\sigma)|k|} = |P|_s.$$

Point (3) follows from (1), (2) and analyticity: we can write

$$P(\theta + X(\theta)) = \sum_{l\in\mathbb{N}} (l!)^{-1}(\partial^l P(\theta))X^l(\theta), \quad \theta \in \mathbb{T}^n_{s-\sigma}$$

$$|P \circ (\mathrm{Id} + X)|_{s-\sigma} \leq \sum_{l\in\mathbb{N}} (l!)^{-1}|\partial^l P|_{s-\sigma}|X|^l_{s-\sigma} \leq |P|_s$$

$$|P \circ (\mathrm{Id} + X) - P|_{s-\sigma} \leq \sum_{l\geq 1} (l!)^{-1}|\partial^l P|_{s-\sigma}||X|^l_{s-\sigma} \leq \sigma^{-1}|X|_{s-\sigma}|P|_s.$$

Point (4) follows from (1), (3) and Taylor formula.

Properties of analytic functions: proof ($n = 1$)

Points (1) and (2):

$$|PQ|_s = \sum_{k\in\mathbb{Z}} \left| \sum_{m\leq k} P_m Q_{k-m} \right| e^{2\pi s|k|} \leq \sum_{k,m} |P_m||Q_{k-m}||e^{2\pi s|m|}|e^{2\pi s|k-m|}$$

$$\sum_{l\in\mathbb{N}} (l!)^{-1}\sigma^l |\partial^l P|_{s-\sigma} \leq \sum_{l\in\mathbb{N}} \sum_{k\in\mathbb{Z}} (l!)^{-1} (\sigma 2\pi |k|)^l |P_k| e^{2\pi(s-\sigma)|k|} = |P|_s.$$

Point (3) follows from (1), (2) and analyticity: we can write

$$P(\theta + X(\theta)) = \sum_{l\in\mathbb{N}} (l!)^{-1} (\partial^l P(\theta)) X^l(\theta), \quad \theta \in \mathbb{T}^n_{s-\sigma}$$

$$|P \circ (\mathrm{Id} + X)|_{s-\sigma} \leq \sum_{l\in\mathbb{N}} (l!)^{-1} |\partial^l P|_{s-\sigma} |X|^l_{s-\sigma} \leq |P|_s$$

$$|P \circ (\mathrm{Id} + X) - P|_{s-\sigma} \leq \sum_{l\geq 1} (l!)^{-1} |\partial^l P|_{s-\sigma} ||X|^l_{s-\sigma} \leq \sigma^{-1} |X|_{s-\sigma} |P|_s.$$

Point (4) follows from (1), (3) and Taylor formula. Point (5) follows from (3), (4) and the contraction principle.

Conjugacy and cohomological equation

Conjugacy and cohomological equation

Given $P : \mathbb{T}^n \to \mathbb{R}^n$, we want to find (U, v) or (V, v) such that

$$(\mathrm{Id} + \mathrm{U}) \circ (F - v) = T_\alpha \circ (\mathrm{Id} + U) \quad \Leftrightarrow \quad U - U \circ (F - v) = P - v.$$

$$(\mathrm{Id} - \mathrm{V}) \circ T_\alpha = (F - v) \circ (\mathrm{Id} - V) \quad \Leftrightarrow \quad V - V \circ (\mathrm{Id} + \alpha) = P \circ (\mathrm{Id} - V) - v.$$

Conjugacy and cohomological equation

Given $P : \mathbb{T}^n \to \mathbb{R}^n$, we want to find (U, v) or (V, v) such that

$$(\mathrm{Id} + \mathrm{U}) \circ (F - v) = T_\alpha \circ (\mathrm{Id} + U) \quad \Leftrightarrow \quad U - U \circ (F - v) = P - v.$$

$$(\mathrm{Id} - \mathrm{V}) \circ T_\alpha = (F - v) \circ (\mathrm{Id} - V) \quad \Leftrightarrow \quad V - V \circ (\mathrm{Id} + \alpha) = P \circ (\mathrm{Id} - V) - v.$$

Since $P \sim \varepsilon$, one expects $U \sim \varepsilon$, $V \sim \varepsilon$, $v \sim \varepsilon$ hence

$$U \circ (F - v) - U \circ (\mathrm{Id} + \alpha) = O(\varepsilon^2), \quad P \circ (\mathrm{Id} - V) - P = O(\varepsilon^2).$$

Conjugacy and cohomological equation

Given $P : \mathbb{T}^n \to \mathbb{R}^n$, we want to find (U, v) or (V, v) such that

$$(\mathrm{Id} + \mathrm{U}) \circ (F - v) = T_\alpha \circ (\mathrm{Id} + U) \quad \Leftrightarrow \quad U - U \circ (F - v) = P - v.$$

$$(\mathrm{Id} - \mathrm{V}) \circ T_\alpha = (F - v) \circ (\mathrm{Id} - V) \quad \Leftrightarrow \quad V - V \circ (\mathrm{Id} + \alpha) = P \circ (\mathrm{Id} - V) - v.$$

Since $P \sim \varepsilon$, one expects $U \sim \varepsilon$, $V \sim \varepsilon$, $v \sim \varepsilon$ hence

$$U \circ (F - v) - U \circ (\mathrm{Id} + \alpha) = O(\varepsilon^2), \quad P \circ (\mathrm{Id} - V) - P = O(\varepsilon^2).$$

The cohomological equation, or linearized conjugacy equation, amounts to find $(\tilde{U}, \tilde{u})$ or $(\tilde{V}, \tilde{v})$ such that

$$\tilde{U} - \tilde{U} \circ (\mathrm{Id} + \alpha) = P - \tilde{u}, \quad \tilde{V} - \tilde{V} \circ (\mathrm{Id} + \alpha) = P - \tilde{v}.$$

Conjugacy and cohomological equation

Given $P : \mathbb{T}^n \to \mathbb{R}^n$, we want to find (U, v) or (V, v) such that

$$(\mathrm{Id} + \mathrm{U}) \circ (F - v) = T_\alpha \circ (\mathrm{Id} + U) \quad \Leftrightarrow \quad U - U \circ (F - v) = P - v.$$

$$(\mathrm{Id} - \mathrm{V}) \circ T_\alpha = (F - v) \circ (\mathrm{Id} - V) \quad \Leftrightarrow \quad V - V \circ (\mathrm{Id} + \alpha) = P \circ (\mathrm{Id} - V) - v.$$

Since $P \sim \varepsilon$, one expects $U \sim \varepsilon$, $V \sim \varepsilon$, $v \sim \varepsilon$ hence

$$U \circ (F - v) - U \circ (\mathrm{Id} + \alpha) = O(\varepsilon^2), \quad P \circ (\mathrm{Id} - V) - P = O(\varepsilon^2).$$

The cohomological equation, or linearized conjugacy equation, amounts to find $(\tilde{U}, \tilde{u})$ or $(\tilde{V}, \tilde{v})$ such that

$$\tilde{U} - \tilde{U} \circ (\mathrm{Id} + \alpha) = P - \tilde{u}, \quad \tilde{V} - \tilde{V} \circ (\mathrm{Id} + \alpha) = P - \tilde{v}.$$

This can be solved in the space of formal Fourier series: $\tilde{u} = [P] = P_0$

$$\tilde{U}_k - e^{2\pi i k \cdot \alpha} \tilde{U}_k = P_k, \quad k \in \mathbb{Z}^n \setminus \{0\}$$

$$\Longleftrightarrow \quad \tilde{U}_k = \frac{P_k}{1 - e^{2\pi i k \cdot \alpha}}, \quad k \in \mathbb{Z}^n \setminus \{0\}.$$

Conjugacy and cohomological equation

Given $P : \mathbb{T}^n \to \mathbb{R}^n$, we want to find (U, v) or (V, v) such that

$$(\mathrm{Id} + \mathrm{U}) \circ (F - v) = T_\alpha \circ (\mathrm{Id} + U) \quad \Leftrightarrow \quad U - U \circ (F - v) = P - v.$$

$$(\mathrm{Id} - \mathrm{V}) \circ T_\alpha = (F - v) \circ (\mathrm{Id} - V) \quad \Leftrightarrow \quad V - V \circ (\mathrm{Id} + \alpha) = P \circ (\mathrm{Id} - V) - v.$$

Since $P \sim \varepsilon$, one expects $U \sim \varepsilon$, $V \sim \varepsilon$, $v \sim \varepsilon$ hence

$$U \circ (F - v) - U \circ (\mathrm{Id} + \alpha) = O(\varepsilon^2), \quad P \circ (\mathrm{Id} - V) - P = O(\varepsilon^2).$$

The cohomological equation, or linearized conjugacy equation, amounts to find $(\tilde{U}, \tilde{u})$ or $(\tilde{V}, \tilde{v})$ such that

$$\tilde{U} - \tilde{U} \circ (\mathrm{Id} + \alpha) = P - \tilde{u}, \quad \tilde{V} - \tilde{V} \circ (\mathrm{Id} + \alpha) = P - \tilde{v}.$$

This can be solved in the space of formal Fourier series: $\tilde{u} = [P] = P_0$

$$\tilde{U}_k - e^{2\pi i k \cdot \alpha} \tilde{U}_k = P_k, \quad k \in \mathbb{Z}^n \setminus \{0\}$$

$$\Longleftrightarrow \quad \tilde{U}_k = \frac{P_k}{1 - e^{2\pi i k \cdot \alpha}}, \quad k \in \mathbb{Z}^n \setminus \{0\}.$$

Solution is unique if $\tilde{U}_0 = 0$.

Conjugacy and cohomological equation

Given $P:\mathbb{T}^n\to\mathbb{R}^n$, we want to find (U,v) or (V,v) such that

$$(\mathrm{Id}+\mathrm{U})\circ(F-v)=T_\alpha\circ(\mathrm{Id}+U)\quad\Leftrightarrow\quad U-U\circ(F-v)=P-v.$$

$$(\mathrm{Id}-\mathrm{V})\circ T_\alpha=(F-v)\circ(\mathrm{Id}-V)\quad\Leftrightarrow\quad V-V\circ(\mathrm{Id}+\alpha)=P\circ(\mathrm{Id}-V)-v.$$

Since $P\sim\varepsilon$, one expects $U\sim\varepsilon$, $V\sim\varepsilon$, $v\sim\varepsilon$ hence

$$U\circ(F-v)-U\circ(\mathrm{Id}+\alpha)=O(\varepsilon^2),\quad P\circ(\mathrm{Id}-V)-P=O(\varepsilon^2).$$

The cohomological equation, or linearized conjugacy equation, amounts to find $(\tilde{U},\tilde{u})$ or $(\tilde{V},\tilde{v})$ such that

$$\tilde{U}-\tilde{U}\circ(\mathrm{Id}+\alpha)=P-\tilde{u},\quad \tilde{V}-\tilde{V}\circ(\mathrm{Id}+\alpha)=P-\tilde{v}.$$

This can be solved in the space of formal Fourier series: $\tilde{u}=[P]=P_0$

$$\tilde{U}_k-e^{2\pi ik\cdot\alpha}\tilde{U}_k=P_k,\quad k\in\mathbb{Z}^n\setminus\{0\}$$

$$\Longleftrightarrow\quad \tilde{U}_k=\frac{P_k}{1-e^{2\pi ik\cdot\alpha}},\quad k\in\mathbb{Z}^n\setminus\{0\}.$$

Solution is unique if $\tilde{U}_0=0$. If P is real ($\overline{P_k}=P_{-k}$) then so is $\tilde{U}$.

Cohomological equation

Cohomological equation

Proposition

For any P real-analytic, there exists a unique couple $(\tilde{U}, \tilde{u})$

$$\tilde{U} - \tilde{U} \circ (\mathrm{Id} + \alpha) = P - \tilde{u}$$

and for any $0 < \sigma \leq s$, setting $\bar{C} = (\tau + 1)!4^{-1}(2\pi)^{-\tau}$,

$$|\tilde{u}| \leq |P|_s, \quad |\tilde{U}|_{s-\sigma} \leq \frac{\bar{C}}{\gamma\sigma^{\tau}}|P|_s, \quad |D\tilde{U}|_{s-\sigma} \leq \frac{\bar{C}}{\gamma\sigma^{\tau+1}}|P|_s.$$

Cohomological equation

Proposition

For any P real-analytic, there exists a unique couple $(\tilde{U}, \tilde{u})$

$$\tilde{U} - \tilde{U} \circ (\mathrm{Id} + \alpha) = P - \tilde{u}$$

and for any $0 < \sigma \leq s$, setting $\bar{C} = (\tau + 1)! 4^{-1} (2\pi)^{-\tau}$,

$$|\tilde{u}| \leq |P|_s, \quad |\tilde{U}|_{s-\sigma} \leq \frac{\bar{C}}{\gamma \sigma^{\tau}} |P|_s, \quad |D\tilde{U}|_{s-\sigma} \leq \frac{\bar{C}}{\gamma \sigma^{\tau+1}} |P|_s.$$

Proof.

Since $|1 - e^{2\pi i k \cdot \alpha}| = 2|\sin(\pi k \cdot \alpha)| \geq 4|k \cdot \alpha|_{\mathbb{Z}} \geq 4\gamma |k|^{-\tau}$:

Cohomological equation

Proposition

For any P real-analytic, there exists a unique couple $(\tilde{U}, \tilde{u})$

$$\tilde{U} - \tilde{U} \circ (\mathrm{Id} + \alpha) = P - \tilde{u}$$

and for any $0 < \sigma \leq s$, setting $\bar{C} = (\tau+1)!4^{-1}(2\pi)^{-\tau}$,

$$|\tilde{u}| \leq |P|_s, \quad |\tilde{U}|_{s-\sigma} \leq \frac{\bar{C}}{\gamma\sigma^{\tau}}|P|_s, \quad |D\tilde{U}|_{s-\sigma} \leq \frac{\bar{C}}{\gamma\sigma^{\tau+1}}|P|_s.$$

Proof.

Since $|1 - e^{2\pi i k\cdot\alpha}| = 2|\sin(\pi k\cdot\alpha)| \geq 4|k\cdot\alpha|_{\mathbb{Z}} \geq 4\gamma|k|^{-\tau}$:

$$\begin{aligned} |\tilde{U}|_{s-\sigma} &\leq \frac{1}{4\gamma}\sum_{k\in\mathbb{Z}^n\setminus\{0\}} |k|^{\tau} e^{-2\pi\sigma|k|}|P_k|e^{2\pi s|k|} \\ &\leq \frac{1}{4\gamma}\left(\sup_{k\in\mathbb{Z}^n\setminus\{0\}} |k|^{\tau} e^{-2\pi\sigma|k|}\right)|P|_r \leq \frac{\tau!}{4\gamma(2\pi\sigma)^{\tau}}|P|_s. \end{aligned}$$

Cohomological equation

Proposition

For any P real-analytic, there exists a unique couple $(\tilde{U}, \tilde{u})$

$$\tilde{U} - \tilde{U} \circ (\mathrm{Id} + \alpha) = P - \tilde{u}$$

and for any $0 < \sigma \leq s$, setting $\bar{C} = (\tau + 1)! 4^{-1} (2\pi)^{-\tau}$,

$$|\tilde{u}| \leq |P|_s, \quad |\tilde{U}|_{s-\sigma} \leq \frac{\bar{C}}{\gamma \sigma^{\tau}} |P|_s, \quad |D\tilde{U}|_{s-\sigma} \leq \frac{\bar{C}}{\gamma \sigma^{\tau+1}} |P|_s.$$

Proof.

Since $|1 - e^{2\pi i k\cdot\alpha}| = 2|\sin(\pi k \cdot \alpha)| \geq 4|k \cdot \alpha|_{\mathbb{Z}} \geq 4\gamma |k|^{-\tau}$:

$$\begin{aligned} |\tilde{U}|_{s-\sigma} &\leq \frac{1}{4\gamma} \sum_{k \in \mathbb{Z}^n \setminus \{0\}} |k|^{\tau} e^{-2\pi\sigma|k|} |P_k| e^{2\pi s|k|} \\ &\leq \frac{1}{4\gamma} \left(\sup_{k \in \mathbb{Z}^n \setminus \{0\}} |k|^{\tau} e^{-2\pi\sigma|k|} \right) |P|_r \leq \frac{\tau!}{4\gamma (2\pi\sigma)^{\tau}} |P|_s. \end{aligned}$$

$$|D\tilde{U}|_{s-\sigma} \leq \frac{1}{4\gamma} \left(\sup_{k \in \mathbb{Z}^n \setminus \{0\}} 2\pi |k|^{\tau+1} e^{-2\pi\sigma|k|} \right) |P|_r \leq \frac{(\tau+1)!}{4\gamma (2\pi\sigma)^{\tau+1}} |P|_s.$$

Local uniqueness of the KAM normal norm

Local uniqueness of the KAM normal norm

For any real-analytic $P : \mathbb{T}^n \to \mathbb{R}^n$ satisfying

$$\varepsilon := |P|_s \leq c_* \gamma s^{\tau+1}, \quad c_* = (4C_*)^{-1} < 1, \quad C_* := \bar{C} n 2^6 16^{\tau+1} > 1$$

assume we have two solutions (V_1, v_1), (V_2, v_2) of

$$V_i - V_i \circ (\mathrm{Id} + \alpha) = P \circ (\mathrm{Id} - V_i) - v_i, \quad |V_i|_{s/2} \leq \frac{C_* \varepsilon}{\gamma s^\tau} \leq s/4, \quad i = 1, 2.$$

Local uniqueness of the KAM normal norm

For any real-analytic $P : \mathbb{T}^n \to \mathbb{R}^n$ satisfying

$$\varepsilon := |P|_s \leq c_* \gamma s^{\tau+1}, \quad c_* = (4C_*)^{-1} < 1, \quad C_* := \bar{C} n 2^6 16^{\tau+1} > 1$$

assume we have two solutions (V_1, v_1), (V_2, v_2) of

$$V_i - V_i \circ (\mathrm{Id} + \alpha) = P \circ (\mathrm{Id} - V_i) - v_i, \quad |V_i|_{s/2} \leq \frac{C_* \varepsilon}{\gamma s^\tau} \leq s/4, \quad i = 1, 2.$$

Let $\hat{V} = V_1 - V_2$, $\hat{v} = \hat{v}_1 - \hat{v}_2$, $\hat{P} = P \circ (\mathrm{Id} - V_1) - P \circ (\mathrm{Id} - V_2)$, then:

$$\hat{V} - \hat{V} \circ (\mathrm{Id} + \alpha) = \hat{P} - \hat{v}, \quad \hat{v} = [\hat{P}].$$

Local uniqueness of the KAM normal norm

For any real-analytic $P : \mathbb{T}^n \to \mathbb{R}^n$ satisfying

$$\varepsilon := |P|_s \leq c_* \gamma s^{\tau+1}, \quad c_* = (4C_*)^{-1} < 1, \quad C_* := \bar{C} n 2^6 16^{\tau+1} > 1$$

assume we have two solutions (V_1, v_1), (V_2, v_2) of

$$V_i - V_i \circ (\mathrm{Id} + \alpha) = P \circ (\mathrm{Id} - V_i) - v_i, \quad |V_i|_{s/2} \leq \frac{C_* \varepsilon}{\gamma s^\tau} \leq s/4, \quad i = 1, 2.$$

Let $\hat{V} = V_1 - V_2$, $\hat{v} = \hat{v}_1 - \hat{v}_2$, $\hat{P} = P \circ (\mathrm{Id} - V_1) - P \circ (\mathrm{Id} - V_2)$, then:

$$\hat{V} - \hat{V} \circ (\mathrm{Id} + \alpha) = \hat{P} - \hat{v}, \quad \hat{v} = [\hat{P}].$$

Using Taylor formula

$$|\hat{P}|_{s/4} \leq |DP|_{s/2} |\hat{V}|_{s/4} \leq 2s^{-1} \varepsilon |\hat{V}|_0, \quad |\hat{P} - [\hat{P}]|_{s/4} \leq |\hat{P}|_{s/4} \leq 2s^{-1} \varepsilon |\hat{V}|_0$$

Local uniqueness of the KAM normal norm

For any real-analytic $P : \mathbb{T}^n \to \mathbb{R}^n$ satisfying

$$\varepsilon := |P|_s \leq c_* \gamma s^{\tau+1}, \quad c_* = (4C_*)^{-1} < 1, \quad C_* := \bar{C} n 2^6 16^{\tau+1} > 1$$

assume we have two solutions (V_1, v_1), (V_2, v_2) of

$$V_i - V_i \circ (\mathrm{Id} + \alpha) = P \circ (\mathrm{Id} - V_i) - v_i, \quad |V_i|_{s/2} \leq \frac{C_* \varepsilon}{\gamma s^\tau} \leq s/4, \quad i = 1, 2.$$

Let $\hat{V} = V_1 - V_2$, $\hat{v} = \hat{v}_1 - \hat{v}_2$, $\hat{P} = P \circ (\mathrm{Id} - V_1) - P \circ (\mathrm{Id} - V_2)$, then:

$$\hat{V} - \hat{V} \circ (\mathrm{Id} + \alpha) = \hat{P} - \hat{v}, \quad \hat{v} = [\hat{P}].$$

Using Taylor formula

$$|\hat{P}|_{s/4} \leq |DP|_{s/2} |\hat{V}|_{s/4} \leq 2s^{-1} \varepsilon |\hat{V}|_0, \quad |\hat{P} - [\hat{P}]|_{s/4} \leq |\hat{P}|_{s/4} \leq 2s^{-1} \varepsilon |\hat{V}|_0$$

and using the cohomological equation ($[\hat{V}] = [V_1] - [V_2] = 0$)

$$|\hat{V}|_0 \leq \frac{4\bar{C}}{\gamma s^\tau} |\hat{P} - [\hat{P}]|_{s/4} \leq \frac{8\bar{C}}{\gamma s^{\tau+1}} \varepsilon |\hat{V}|_0 \leq \frac{C_*}{\gamma s^{\tau+1}} \varepsilon |\hat{V}|_0 \leq |\hat{V}|_0/4.$$

Local uniqueness of the KAM normal norm

For any real-analytic $P : \mathbb{T}^n \to \mathbb{R}^n$ satisfying

$$\varepsilon := |P|_s \leq c_* \gamma s^{\tau+1}, \quad c_* = (4C_*)^{-1} < 1, \quad C_* := \bar{C} n 2^6 16^{\tau+1} > 1$$

assume we have two solutions (V_1, v_1), (V_2, v_2) of

$$V_i - V_i \circ (\mathrm{Id} + \alpha) = P \circ (\mathrm{Id} - V_i) - v_i, \quad |V_i|_{s/2} \leq \frac{C_* \varepsilon}{\gamma s^\tau} \leq s/4, \quad i = 1, 2.$$

Let $\hat{V} = V_1 - V_2$, $\hat{v} = \hat{v}_1 - \hat{v}_2$, $\hat{P} = P \circ (\mathrm{Id} - V_1) - P \circ (\mathrm{Id} - V_2)$, then:

$$\hat{V} - \hat{V} \circ (\mathrm{Id} + \alpha) = \hat{P} - \hat{v}, \quad \hat{v} = [\hat{P}].$$

Using Taylor formula

$$|\hat{P}|_{s/4} \leq |DP|_{s/2} |\hat{V}|_{s/4} \leq 2s^{-1} \varepsilon |\hat{V}|_0, \quad |\hat{P} - [\hat{P}]|_{s/4} \leq |\hat{P}|_{s/4} \leq 2s^{-1} \varepsilon |\hat{V}|_0$$

and using the cohomological equation ($[\hat{V}] = [V_1] - [V_2] = 0$)

$$|\hat{V}|_0 \leq \frac{4\bar{C}}{\gamma s^\tau} |\hat{P} - [\hat{P}]|_{s/4} \leq \frac{8\bar{C}}{\gamma s^{\tau+1}} \varepsilon |\hat{V}|_0 \leq \frac{C_*}{\gamma s^{\tau+1}} \varepsilon |\hat{V}|_0 \leq |\hat{V}|_0 / 4.$$

Hence $\hat{V} = 0 \Rightarrow V_1 = V_2 \Rightarrow \hat{P} = 0 \Rightarrow \hat{v} = 0 \Rightarrow v_1 = v_2$.

Failure of the Picard method (contraction principle)

Failure of the Picard method (contraction principle)

Let $\mathcal{A}_s$ the space of real-analytic vector fields, $\mathcal{A}_s^0$ the subspace with zero-average, and given $\alpha \in K_\gamma^\tau$, consider the bounded linear operator

$$\mathcal{L}_\alpha : \mathcal{A}_s \to \mathcal{A}_s^0 \quad \mathcal{L}_\alpha(V) = V - \mathrm{V} \circ (\mathrm{Id} + \alpha).$$

Failure of the Picard method (contraction principle)

Let $\mathcal{A}_s$ the space of real-analytic vector fields, $\mathcal{A}_s^0$ the subspace with zero-average, and given $\alpha \in K_\gamma^\tau$, consider the bounded linear operator

$$\mathcal{L}_\alpha : \mathcal{A}_s \to \mathcal{A}_s^0 \quad \mathcal{L}_\alpha(V) = V - \mathrm{V} \circ (\mathrm{Id} + \alpha).$$

Now given $P \in \mathcal{A}_s$, consider the bounded non-linear operator

$$\mathcal{N}_P(V) = P \circ (\mathrm{Id} - V) - v, \quad v = v(V, P) := [P \circ (\mathrm{Id} - V)].$$

Failure of the Picard method (contraction principle)

Let $\mathcal{A}_s$ the space of real-analytic vector fields, $\mathcal{A}_s^0$ the subspace with zero-average, and given $\alpha \in K_\gamma^\tau$, consider the bounded linear operator

$$\mathcal{L}_\alpha : \mathcal{A}_s \to \mathcal{A}_s^0 \quad \mathcal{L}_\alpha(V) = V - \mathrm{V} \circ (\mathrm{Id} + \alpha).$$

Now given $P \in \mathcal{A}_s$, consider the bounded non-linear operator

$$\mathcal{N}_P(V) = P \circ (\mathrm{Id} - V) - v, \quad v = v(V, P) := [P \circ (\mathrm{Id} - V)].$$

Then the conjugacy equation can be written as

$$\mathcal{L}_\alpha(V) = \mathcal{N}_P(V) \Longleftrightarrow V = \mathcal{F}_{\alpha,P}(V), \quad \mathcal{F}_{\alpha,P} := \mathcal{L}_\alpha^{-1} \circ \mathcal{N}_P.$$

Failure of the Picard method (contraction principle)

Let $\mathcal{A}_s$ the space of real-analytic vector fields, $\mathcal{A}_s^0$ the subspace with zero-average, and given $\alpha \in K_\gamma^\tau$, consider the bounded linear operator

$$\mathcal{L}_\alpha : \mathcal{A}_s \to \mathcal{A}_s^0 \quad \mathcal{L}_\alpha(V) = V - \mathrm{V} \circ (\mathrm{Id} + \alpha).$$

Now given $P \in \mathcal{A}_s$, consider the bounded non-linear operator

$$\mathcal{N}_P(V) = P \circ (\mathrm{Id} - V) - v, \quad v = v(V, P) := [P \circ (\mathrm{Id} - V)].$$

Then the conjugacy equation can be written as

$$\mathcal{L}_\alpha(V) = \mathcal{N}_P(V) \Longleftrightarrow V = \mathcal{F}_{\alpha,P}(V), \quad \mathcal{F}_{\alpha,P} := \mathcal{L}_\alpha^{-1} \circ \mathcal{N}_P.$$

Let $\varepsilon := |P|_s$ and $\mathcal{B}_\varepsilon := \{V \in \mathcal{A}_{s/2}^0 \mid |V|_{s/2} \leq \varepsilon\}$.

Failure of the Picard method (contraction principle)

Let $\mathcal{A}_s$ the space of real-analytic vector fields, $\mathcal{A}_s^0$ the subspace with zero-average, and given $\alpha \in K_\gamma^\tau$, consider the bounded linear operator

$$\mathcal{L}_\alpha : \mathcal{A}_s \to \mathcal{A}_s^0 \quad \mathcal{L}_\alpha(V) = V - \mathrm{V} \circ (\mathrm{Id} + \alpha).$$

Now given $P \in \mathcal{A}_s$, consider the bounded non-linear operator

$$\mathcal{N}_P(V) = P \circ (\mathrm{Id} - V) - v, \quad v = v(V, P) := [P \circ (\mathrm{Id} - V)].$$

Then the conjugacy equation can be written as

$$\mathcal{L}_\alpha(V) = \mathcal{N}_P(V) \Longleftrightarrow V = \mathcal{F}_{\alpha,P}(V), \quad \mathcal{F}_{\alpha,P} := \mathcal{L}_\alpha^{-1} \circ \mathcal{N}_P.$$

Let $\varepsilon := |P|_s$ and $\mathcal{B}_\varepsilon := \{V \in \mathcal{A}_{s/2}^0 \mid |V|_{s/2} \leq \varepsilon\}$. Then $\mathcal{N}_P$ is a contraction of $\mathcal{B}_\varepsilon$ for $\varepsilon < s/4$.

Failure of the Picard method (contraction principle)

Let $\mathcal{A}_s$ the space of real-analytic vector fields, $\mathcal{A}_s^0$ the subspace with zero-average, and given $\alpha \in K_\gamma^\tau$, consider the bounded linear operator

$$\mathcal{L}_\alpha : \mathcal{A}_s \to \mathcal{A}_s^0 \quad \mathcal{L}_\alpha(V) = V - \mathrm{V} \circ (\mathrm{Id} + \alpha).$$

Now given $P \in \mathcal{A}_s$, consider the bounded non-linear operator

$$\mathcal{N}_P(V) = P \circ (\mathrm{Id} - V) - v, \quad v = v(V, P) := [P \circ (\mathrm{Id} - V)].$$

Then the conjugacy equation can be written as

$$\mathcal{L}_\alpha(V) = \mathcal{N}_P(V) \Longleftrightarrow V = \mathcal{F}_{\alpha,P}(V), \quad \mathcal{F}_{\alpha,P} := \mathcal{L}_\alpha^{-1} \circ \mathcal{N}_P.$$

Let $\varepsilon := |P|_s$ and $\mathcal{B}_\varepsilon := \{V \in \mathcal{A}_{s/2}^0 \mid |V|_{s/2} \leq \varepsilon\}$. Then $\mathcal{N}_P$ is a contraction of $\mathcal{B}_\varepsilon$ for $\varepsilon < s/4$. If $\mathcal{L}_\alpha^{-1}$ were a bounded operator

$$|\mathcal{L}_\alpha^{-1}(X)|_{s/2} \leq C|X|_{s/2}, \quad X = \mathcal{N}_P(V), \quad C \geq 1$$

then $\mathcal{F}_{\alpha,P}$ would be a contraction of $\mathcal{B}_{C\varepsilon}$ for $C\varepsilon < s/4$.

Failure of the Picard method (contraction principle)

Let $\mathcal{A}_s$ the space of real-analytic vector fields, $\mathcal{A}_s^0$ the subspace with zero-average, and given $\alpha \in K_\gamma^\tau$, consider the bounded linear operator

$$\mathcal{L}_\alpha : \mathcal{A}_s \to \mathcal{A}_s^0 \quad \mathcal{L}_\alpha(V) = V - \mathrm{V} \circ (\mathrm{Id} + \alpha).$$

Now given $P \in \mathcal{A}_s$, consider the bounded non-linear operator

$$\mathcal{N}_P(V) = P \circ (\mathrm{Id} - V) - v, \quad v = v(V, P) := [P \circ (\mathrm{Id} - V)].$$

Then the conjugacy equation can be written as

$$\mathcal{L}_\alpha(V) = \mathcal{N}_P(V) \Longleftrightarrow V = \mathcal{F}_{\alpha,P}(V), \quad \mathcal{F}_{\alpha,P} := \mathcal{L}_\alpha^{-1} \circ \mathcal{N}_P.$$

Let $\varepsilon := |P|_s$ and $\mathcal{B}_\varepsilon := \{V \in \mathcal{A}_{s/2}^0 \mid |V|_{s/2} \leq \varepsilon\}$. Then $\mathcal{N}_P$ is a contraction of $\mathcal{B}_\varepsilon$ for $\varepsilon < s/4$. If $\mathcal{L}_\alpha^{-1}$ were a bounded operator

$$|\mathcal{L}_\alpha^{-1}(X)|_{s/2} \leq C|X|_{s/2}, \quad X = \mathcal{N}_P(V), \quad C \geq 1$$

then $\mathcal{F}_{\alpha,P}$ would be a contraction of $\mathcal{B}_{C\varepsilon}$ for $C\varepsilon < s/4$. Unfortunately

$$|\mathcal{L}_\alpha^{-1}(X)|_{s/2-\sigma} \leq C(\sigma)|X|_{s/2}, \quad 0 < \sigma \leq /2, \quad \lim_{\sigma \to 0} C(\sigma) = +\infty.$$

Newton method: quadratic remainder

Newton method: quadratic remainder

The basic idea is to use $\mathrm{Id} + \tilde{U}$, where $\tilde{U}$ solves the cohomological equation, to conjugate $F = T_\alpha + P$ to $F_+ = T_\alpha + P_+$ with $|P_+| \ll |P|$.

Newton method: quadratic remainder

The basic idea is to use $\mathrm{Id} + \tilde{U}$, where $\tilde{U}$ solves the cohomological equation, to conjugate $F = T_\alpha + P$ to $F_+ = T_\alpha + P_+$ with $|P_+| \ll |P|$.

$$F_+ = (\mathrm{Id} + \tilde{U}) \circ F \circ (\mathrm{Id} + \tilde{U})^{-1} = T_\alpha + \tilde{u} + R_+ = T_\alpha + [P] + R_+$$

Newton method: quadratic remainder

The basic idea is to use $\mathrm{Id} + \tilde{U}$, where $\tilde{U}$ solves the cohomological equation, to conjugate $F = T_\alpha + P$ to $F_+ = T_\alpha + P_+$ with $|P_+| \ll |P|$.

$$F_+ = (\mathrm{Id} + \tilde{U}) \circ F \circ (\mathrm{Id} + \tilde{U})^{-1} = T_\alpha + \tilde{u} + R_+ = T_\alpha + [P] + R_+$$

$$R_+ = \left(\tilde{U} \circ (\mathrm{Id} + \alpha) - \tilde{U} \circ (\mathrm{Id} + \alpha) \circ (\mathrm{Id} + P)\right) \circ (\mathrm{Id} + \tilde{U})^{-1}.$$

Newton method: quadratic remainder

The basic idea is to use $\mathrm{Id} + \tilde{U}$, where $\tilde{U}$ solves the cohomological equation, to conjugate $F = T_\alpha + P$ to $F_+ = T_\alpha + P_+$ with $|P_+| \ll |P|$.

$$F_+ = (\mathrm{Id} + \tilde{U}) \circ F \circ (\mathrm{Id} + \tilde{U})^{-1} = T_\alpha + \tilde{u} + R_+ = T_\alpha + [P] + R_+$$

$$R_+ = \left(\tilde{U} \circ (\mathrm{Id} + \alpha) - \tilde{U} \circ (\mathrm{Id} + \alpha) \circ (\mathrm{Id} + P) \right) \circ (\mathrm{Id} + \tilde{U})^{-1}.$$

Problem: $|P_+| \sim |\tilde{u}| + |R_+| \sim \varepsilon + \varepsilon^2 \sim \varepsilon$.

Newton method: quadratic remainder

The basic idea is to use $\mathrm{Id} + \tilde{U}$, where $\tilde{U}$ solves the cohomological equation, to conjugate $F = T_\alpha + P$ to $F_+ = T_\alpha + P_+$ with $|P_+| \ll |P|$.

$$F_+ = (\mathrm{Id} + \tilde{U}) \circ F \circ (\mathrm{Id} + \tilde{U})^{-1} = T_\alpha + \tilde{u} + R_+ = T_\alpha + [P] + R_+$$

$$R_+ = \left(\tilde{U} \circ (\mathrm{Id} + \alpha) - \tilde{U} \circ (\mathrm{Id} + \alpha) \circ (\mathrm{Id} + P)\right) \circ (\mathrm{Id} + \tilde{U})^{-1}.$$

Problem: $|P_+| \sim |\tilde{u}| + |R_+| \sim \varepsilon + \varepsilon^2 \sim \varepsilon$. Temporary solution: forget about $\tilde{u}$, assume $P_+ = R_+$ (this will be achieved using $v = \tilde{u} + O(\varepsilon^2)$).

Newton method: quadratic remainder

The basic idea is to use $\mathrm{Id} + \tilde{U}$, where $\tilde{U}$ solves the cohomological equation, to conjugate $F = T_\alpha + P$ to $F_+ = T_\alpha + P_+$ with $|P_+| \ll |P|$.

$$F_+ = (\mathrm{Id} + \tilde{U}) \circ F \circ (\mathrm{Id} + \tilde{U})^{-1} = T_\alpha + \tilde{u} + R_+ = T_\alpha + [P] + R_+$$

$$R_+ = \left(\tilde{U} \circ (\mathrm{Id} + \alpha) - \tilde{U} \circ (\mathrm{Id} + \alpha) \circ (\mathrm{Id} + P)\right) \circ (\mathrm{Id} + \tilde{U})^{-1}.$$

Problem: $|P_+| \sim |\tilde{u}| + |R_+| \sim \varepsilon + \varepsilon^2 \sim \varepsilon$. Temporary solution: forget about $\tilde{u}$, assume $P_+ = R_+$ (this will be achieved using $v = \tilde{u} + O(\varepsilon^2)$). Let $0 < \sigma \leq s$, one can estimate

$$|R_+|_{s-2\sigma} \leq |D\tilde{U}|_{s-\sigma}|P|_s \leq \frac{\bar{C}\varepsilon^2}{\gamma\sigma^{\tau+1}} := \bar{C}(\sigma)\varepsilon^2$$

Newton method: quadratic remainder

The basic idea is to use $\mathrm{Id} + \tilde{U}$, where $\tilde{U}$ solves the cohomological equation, to conjugate $F = T_\alpha + P$ to $F_+ = T_\alpha + P_+$ with $|P_+| \ll |P|$.

$$F_+ = (\mathrm{Id} + \tilde{U}) \circ F \circ (\mathrm{Id} + \tilde{U})^{-1} = T_\alpha + \tilde{u} + R_+ = T_\alpha + [P] + R_+$$

$$R_+ = \Big(\tilde{U} \circ (\mathrm{Id} + \alpha) - \tilde{U} \circ (\mathrm{Id} + \alpha) \circ (\mathrm{Id} + P)\Big) \circ (\mathrm{Id} + \tilde{U})^{-1}.$$

Problem: $|P_+| \sim |\tilde{u}| + |R_+| \sim \varepsilon + \varepsilon^2 \sim \varepsilon$. Temporary solution: forget about $\tilde{u}$, assume $P_+ = R_+$ (this will be achieved using $v = \tilde{u} + O(\varepsilon^2)$). Let $0 < \sigma \leq s$, one can estimate

$$|R_+|_{s-2\sigma} \leq |D\tilde{U}|_{s-\sigma} |P|_s \leq \frac{\bar{C}\varepsilon^2}{\gamma\sigma^{\tau+1}} := \bar{C}(\sigma)\varepsilon^2$$

(assuming $\bar{C}(\sigma) \leq 1/\sigma$) and assuming ε is small enough so that

$$\bar{C}(\sigma)\varepsilon < 1/2 \quad \Longrightarrow \quad |P|_s \leq |\tilde{U}|_{s-\sigma} < \sigma/2, \quad |D\tilde{U}|_{s-\sigma} < 1/2.$$

Newton method: iteration

Newton method: iteration

For σ to be chosen, we will require $\bar{C}(\sigma)\varepsilon \leq \kappa$ for some $\kappa < 1/2$.

Newton method: iteration

For σ to be chosen, we will require $\bar{C}(\sigma)\varepsilon \leq \kappa$ for some $\kappa < 1/2$. Set $P_1 = P_+$, $s_1 = s - 2\sigma$, then $F = T_\alpha + P$ is conjugated to $F_1 = T_\alpha + P_1$ with $|P_1|_{s_1} \leq \bar{C}(\sigma)\varepsilon^2 \leq \kappa\varepsilon$.

Newton method: iteration

For σ to be chosen, we will require $\bar{C}(\sigma)\varepsilon \leq \kappa$ for some $\kappa < 1/2$. Set $P_1 = P_+$, $s_1 = s - 2\sigma$, then $F = T_\alpha + P$ is conjugated to $F_1 = T_\alpha + P_1$ with $|P_1|_{s_1} \leq \bar{C}(\sigma)\varepsilon^2 \leq \kappa\varepsilon$. To proceed by induction, choose

$$\sigma_j := 2^{-j}\sigma, \quad s_0 := s, \quad s_{j+1} := s_j - 2\sigma_j, \quad \varepsilon_j := \kappa^j \varepsilon$$

Newton method: iteration

For σ to be chosen, we will require $\bar{C}(\sigma)\varepsilon \leq \kappa$ for some $\kappa < 1/2$. Set $P_1 = P_+$, $s_1 = s - 2\sigma$, then $F = T_\alpha + P$ is conjugated to $F_1 = T_\alpha + P_1$ with $|P_1|_{s_1} \leq \bar{C}(\sigma)\varepsilon^2 \leq \kappa\varepsilon$. To proceed by induction, choose

$$\sigma_j := 2^{-j}\sigma, \quad s_0 := s, \quad s_{j+1} := s_j - 2\sigma_j, \quad \varepsilon_j := \kappa^j \varepsilon$$

then $F = T_\alpha + P$ is conjugated to $F_j = T_\alpha + P_j$ with $|P_j|_{s_j} \leq \varepsilon_j$ if,

$$\bar{C}(\sigma_j)\varepsilon_j = (\kappa/2^{\tau+1})^j \bar{C}(\sigma_0)\varepsilon_0 < 1/2 \impliedby \bar{C}(\sigma_0)\varepsilon_0 \leq \kappa \leq 2^{-(\tau+1)}.$$

Newton method: iteration

For σ to be chosen, we will require $\bar{C}(\sigma)\varepsilon \leq \kappa$ for some $\kappa < 1/2$. Set $P_1 = P_+$, $s_1 = s - 2\sigma$, then $F = T_\alpha + P$ is conjugated to $F_1 = T_\alpha + P_1$ with $|P_1|_{s_1} \leq \bar{C}(\sigma)\varepsilon^2 \leq \kappa\varepsilon$. To proceed by induction, choose

$$\sigma_j := 2^{-j}\sigma, \quad s_0 := s, \quad s_{j+1} := s_j - 2\sigma_j, \quad \varepsilon_j := \kappa^j \varepsilon$$

then $F = T_\alpha + P$ is conjugated to $F_j = T_\alpha + P_j$ with $|P_j|_{s_j} \leq \varepsilon_j$ if,

$$\bar{C}(\sigma_j)\varepsilon_j = (\kappa/2^{\tau+1})^j \bar{C}(\sigma_0)\varepsilon_0 < 1/2 \Longleftarrow \bar{C}(\sigma_0)\varepsilon_0 \leq \kappa \leq 2^{-(\tau+1)}.$$

Now choose $\sigma = \sigma_0 = s/8$ so that $s_j \to s - 4\sigma \geq s/2$ (and $\bar{C}(\sigma) \leq 1/\sigma$ since $s \leq 1$) and choose $\kappa := 2^{-(\tau+2)}$

Newton method: iteration

For σ to be chosen, we will require $\bar{C}(\sigma)\varepsilon \leq \kappa$ for some $\kappa < 1/2$. Set $P_1 = P_+$, $s_1 = s - 2\sigma$, then $F = T_\alpha + P$ is conjugated to $F_1 = T_\alpha + P_1$ with $|P_1|_{s_1} \leq \bar{C}(\sigma)\varepsilon^2 \leq \kappa\varepsilon$. To proceed by induction, choose

$$\sigma_j := 2^{-j}\sigma, \quad s_0 := s, \quad s_{j+1} := s_j - 2\sigma_j, \quad \varepsilon_j := \kappa^j \varepsilon$$

then $F = T_\alpha + P$ is conjugated to $F_j = T_\alpha + P_j$ with $|P_j|_{s_j} \leq \varepsilon_j$ if,

$$\bar{C}(\sigma_j)\varepsilon_j = (\kappa/2^{\tau+1})^j \bar{C}(\sigma_0)\varepsilon_0 < 1/2 \Longleftarrow \bar{C}(\sigma_0)\varepsilon_0 \leq \kappa \leq 2^{-(\tau+1)}.$$

Now choose $\sigma = \sigma_0 = s/8$ so that $s_j \to s - 4\sigma \geq s/2$ (and $\bar{C}(\sigma) \leq 1/\sigma$ since $s \leq 1$) and choose $\kappa := 2^{-(\tau+2)}$ so that

$$\bar{C}(\sigma_0)\varepsilon_0 = \bar{C}(s/8)\varepsilon \leq 2^{-(\tau+2)} \Longrightarrow \bar{C}(\sigma_j)\varepsilon_j = 2^{-j}\bar{C}(\sigma_0)\varepsilon_0$$

and the infinite composition $\mathrm{Id} + U = \cdots \circ (\mathrm{Id} + \tilde{U}_1) \circ (\mathrm{Id} + \tilde{U}_0)$ converges

$$|DU|_{s/2} \leq 2\sum_{j\in\mathbb{N}} |D\tilde{U}_j|_{s/2} \leq 2\sum_{j\in\mathbb{N}} \bar{C}(\sigma_j)\varepsilon_j \leq 4\bar{C}(\sigma_0)\varepsilon_0.$$

Newton method: iteration

For σ to be chosen, we will require $\bar{C}(\sigma)\varepsilon \leq \kappa$ for some $\kappa < 1/2$. Set $P_1 = P_+$, $s_1 = s - 2\sigma$, then $F = T_\alpha + P$ is conjugated to $F_1 = T_\alpha + P_1$ with $|P_1|_{s_1} \leq \bar{C}(\sigma)\varepsilon^2 \leq \kappa\varepsilon$. To proceed by induction, choose

$$\sigma_j := 2^{-j}\sigma, \quad s_0 := s, \quad s_{j+1} := s_j - 2\sigma_j, \quad \varepsilon_j := \kappa^j \varepsilon$$

then $F = T_\alpha + P$ is conjugated to $F_j = T_\alpha + P_j$ with $|P_j|_{s_j} \leq \varepsilon_j$ if,

$$\bar{C}(\sigma_j)\varepsilon_j = (\kappa/2^{\tau+1})^j \bar{C}(\sigma_0)\varepsilon_0 < 1/2 \Longleftarrow \bar{C}(\sigma_0)\varepsilon_0 \leq \kappa \leq 2^{-(\tau+1)}.$$

Now choose $\sigma = \sigma_0 = s/8$ so that $s_j \to s - 4\sigma \geq s/2$ (and $\bar{C}(\sigma) \leq 1/\sigma$ since $s \leq 1$) and choose $\kappa := 2^{-(\tau+2)}$ so that

$$\bar{C}(\sigma_0)\varepsilon_0 = \bar{C}(s/8)\varepsilon \leq 2^{-(\tau+2)} \Longrightarrow \bar{C}(\sigma_j)\varepsilon_j = 2^{-j}\bar{C}(\sigma_0)\varepsilon_0$$

and the infinite composition $\mathrm{Id} + U = \cdots \circ (\mathrm{Id} + \tilde{U}_1) \circ (\mathrm{Id} + \tilde{U}_0)$ converges

$$|DU|_{s/2} \leq 2\sum_{j\in\mathbb{N}} |D\tilde{U}_j|_{s/2} \leq 2\sum_{j\in\mathbb{N}} \bar{C}(\sigma_j)\varepsilon_j \leq 4\bar{C}(\sigma_0)\varepsilon_0.$$

Actually the sequence $|P_j|_{s_j} \to 0$ “essentially” quadratically.