KAM normal form IV

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$$P(\theta, a) = P_a(\theta) = \sum_{k \in \mathbb{Z}^n} P_k(a) e^{2\pi i k \cdot \theta}, \quad P_k(a) : K_r \to \mathbb{C}^n.$$

with the norm

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Theorem (Arnold)

Fix $\gamma > 0$, $\tau > n$ and $0 < s \leq 1$, and define

$$\sigma := s/8, \quad C(\sigma) := \frac{\bar{C}}{\gamma \sigma^{\tau+1}}, \quad \kappa := 2^{\tau+2}, \quad r := \frac{\kappa}{2C(\sigma)}.$$

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Then for any $\alpha \in K$ and any real-analytic $P : \mathbb{T}^n \times K_r \to \mathbb{R}^n$ satisfying

$$\varepsilon := |P|_{s,r} \leq \frac{r}{64n}$$

there exist a unique couple (U, v), where $U : \mathbb{T}^n \to \mathbb{R}^n$ is real-analytic with zero average and $v \in \mathbb{R}^n$, such that for $F = T_\alpha + P$, $\Phi = \mathrm{Id} + U$ and $\Psi = \Phi^{-1} = \mathrm{Id} - V$, we have

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with the estimates

$$|V|_{s/2} \leq |U|_{s/2} \leq 2\sigma C(\sigma)\varepsilon \leq s/4, \quad |DV|_{s/2} \leq 2|DU|_{s/2} \leq 8C(\sigma)\varepsilon \leq 1/2,$$

$$|v| \leq 2\varepsilon, \quad \mathrm{Lip}_\alpha(v) \leq \frac{32n\varepsilon}{r} \leq 1/2.$$

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$$\tilde{U}_a - \tilde{U}_a \circ (\mathrm{Id} + \alpha) = \hat{P}_a - [\hat{P}_a] = P_a - [P_a]$$

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For $u : K_r \to \mathbb{C}^n$ C^1, let us define

$$\mathrm{Lip}(u)_r := \sup_{x \neq y \in K_r} \frac{|u(x) - u(y)|}{|x - y|}.$$

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$$\frac{|u(x) - u(y)|}{|x - y|} < 8r^{-1}||u||_r.$$

□

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(1) Product. $p, q : K_r \to \mathbb{C}$, $||pq||_r \leq ||p||_r ||q||_r$

(2) Derivative. $p : K_r \to \mathbb{C}^n$ and $0 < \rho \leq r$

$$\sup_{l \in \mathbb{N}^n} \frac{\rho^{|l|}}{l!} ||\partial^l p||_r \leq ||p||_r, \quad ||Dp||_{r-\rho} := \sum_{|l|=1} ||\partial^l p||_{r-\rho} \leq n\rho^{-1} ||p||_r.$$

(3) Composition. $0 < \rho \leq r$, $u : K_{r-\rho} \to \mathbb{C}^n$, $||u||_{r-\rho} \leq \rho$

$$||p \circ (\mathrm{Id} + u)||_{r-\rho} \leq ||p||_r.$$

(4) Taylor. $0 < \rho \leq r$, $u_1, u_2 : \mathbb{T}^n_{r-\rho} \to \mathbb{C}^n$, $||u_i||_{r-\rho} \leq \rho$

$$||p \circ (\mathrm{Id} + u_1) - p \circ (\mathrm{Id} + u_2)||_{r-\rho} \leq \mathrm{Lip}(p)_r ||u_1 - u_2||_{r-2\rho}.$$

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(5) Inverse. $||u||_r \leq r/16n \Rightarrow ||Du||_{3r/4} \leq 1/4 \Rightarrow \mathrm{Lip}(u)_{r/2} \leq 1/2$, then $\varphi = \mathrm{Id} + u : K_{r/2} \to \mathbb{C}^n$ is an analytic embedding such that $K_{r/4} \subseteq \varphi(K_{r/2}) \subseteq K_r$ and $\varphi^{-1} = \mathrm{Id} - v$ with

$$||v||_{r/4} \leq ||u||_r, \quad ||Dv||_{r/4} \leq ||Du||_{3r/4}(1 - ||Du||_{3r/4})^{-1}.$$

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(1) Product. $P, Q : \mathbb{T}^n_{s,r} \to \mathbb{C}$, $|PQ|_{s,r} \leq |P|_{s,r}|Q|_{s,r}$

(2) Derivative. $P : \mathbb{T}^n_{s,r} \to \mathbb{C}^n$ and $0 < \sigma \leq s$

$$\sum_{l \in \mathbb{N}^n} \frac{\sigma^{|l|}}{l!} |\partial^l P|_{s-\sigma,r} \leq |P|_{s,r}, \quad |DP|_{s-\sigma} := \sum_{|l|=1} |\partial^l P|_{s-\sigma,r} \leq \sigma^{-1}|P|_{s,r}.$$

(3) Composition. $0 < \sigma \leq s$, $U : \mathbb{T}^n_{s-\sigma,r} \to \mathbb{C}^n$, $|U|_{s-\sigma,r} \leq \sigma$

$$|P \circ (\mathrm{Id} + U)|_{s-\sigma,r} \leq |P|_{s,r}, \quad |P \circ (\mathrm{Id} + U) - P|_{s-\sigma,r} \leq \sigma^{-1}|U|_{s-\sigma,r}|P|_{s,r}.$$

(4) Taylor. $0 < \sigma \leq s$, $U_1, U_2 : \mathbb{T}^n_{s-\sigma,r} \to \mathbb{C}^n$, $|U_i|_{s-\sigma,r} \leq \sigma$

$$|P \circ (\mathrm{Id} + U_1) - P \circ (\mathrm{Id} + U_2)|_{s-\sigma,r} \leq |DP|_{s,r}|U_1 - U_2|_{s-\sigma}.$$

(5) Inverse. $0 < 2\sigma \leq s$, $|U|_{s-\sigma,r} \leq \sigma$, $|DU|_{s-\sigma,r} < 1$, then $\Phi(\theta, a) = \theta + U(\theta, a) : \mathbb{T}^n_{s-\sigma} \to \mathbb{C}^n$ is an analytic embedding such that $\mathbb{T}^n_{s-2\sigma} \subseteq \Phi(\mathbb{T}^n_{s-\sigma}) \subseteq \mathbb{T}^n_s$ and $\Phi^{-1} = \mathrm{Id} - V$ with

$$|V|_{s-2\sigma,r} \leq |U|_{s-\sigma,r}, \quad |DV|_{s-2\sigma,r} \leq |DU|_{s-\sigma,r}(1 - |DU|_{s-\sigma,r})^{-1}.$$

KAM step

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Let $F_a = T_a + P_a$ *with* $a \in K_r$, $P_a : \mathbb{T}^n_s \to \mathbb{C}^n$, $0 < 2\sigma < s$ *and assume*

$$\varepsilon := |P|_{s,r} \leq \frac{r}{64n}, \quad r = \frac{\kappa}{2C(\sigma)}, \quad C(\sigma) = \frac{\bar{C}}{\gamma\sigma^{\tau+1}}.$$

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Choice of sequences

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Recall $\bar{C} = (\tau + 1)! 4^{-1} (2\pi)^{-\tau}$ and $\kappa = 2^{-(\tau+2)}$, then set

$$\varepsilon_0 = \varepsilon, \quad s_0 = s, \quad \sigma_0 = s/8, \quad \bar{C}(\sigma_0) = \frac{\bar{C}}{\gamma \sigma_0^{\tau+1}}, \quad r_0 = r = \frac{\kappa}{2C(\sigma_0)}.$$

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Choose $\sigma_j = 2^{-j}\sigma_0$, define $s_{j+1} = s_j - 2\sigma_j$, $\hat{s}_{j+1} = \hat{s}_j - \sigma_j$, $\varepsilon_j = \kappa^j\varepsilon_0$

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Then $\varepsilon_j \to 0$, $s_j \to s/2$, $\hat{s}_j \to 3s/4$, $\bar{C}(\sigma_j) \to +\infty$, $r_j \to 0$ and

$$\sum_{j\in\mathbb{N}} \bar{C}(\sigma_j)\varepsilon_j \leq 2\bar{C}(\sigma_0)\varepsilon_0, \quad \sum_{j\in\mathbb{N}} \varepsilon_j/r_j \leq 2\varepsilon_0/r_0.$$

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The final smallness assumption is $\varepsilon \le r/64n$, which implies $\varepsilon_j \le r_j/64n$ and allow to apply the KAM step inductively.

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$$F_\alpha^j \circ \Phi_\alpha^j = \Phi_\alpha^j \circ F_{\psi^j(\alpha)} \Longrightarrow T_\alpha \circ \Phi_\alpha = \Phi_\alpha \circ F_{\psi(\alpha)}, \quad \alpha \in K.$$

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Convergence of v^j (resp. U^j) amounts to convergence (in norm) of $\sum_j (v^{j+1} - v^j)$ in $C^0(K, \mathbb{R}^n)$ (resp. $\sum_j U^{i+1} - U^i$ in $\mathcal{A}(\mathbb{T}^n_{s/2}, \mathbb{C}^n)$).

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Since $|P^j|_{s_j,r_j} \leq \varepsilon_j$, for any $\alpha \in K$, $F^j_\alpha = T_\alpha + P^j_\alpha \to T_\alpha$. Assuming $\psi^j = \mathrm{Id} - v^j \to \psi$ on K and $\Phi^j = \mathrm{Id} + U^j \to \Phi$ on $\mathbb{T}^n_{s/2} \times K$, then

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Convergence of v^j (resp. U^j) amounts to convergence (in norm) of $\sum_j (v^{j+1} - v^j)$ in $C^0(K, \mathbb{R}^n)$ (resp. $\sum_j U^{i+1} - U^i$ in $\mathcal{A}(\mathbb{T}^n_{s/2}, \mathbb{C}^n)$).

$$v^{j+1} - v^j = \tilde{v}^{j+1} + [v^j \circ (\mathrm{Id} - \tilde{v}^{j+1}) - v^j], \quad U^{j+1} - U^j = \tilde{U}^{j+1}_{\tilde{\psi}^j} \circ (\mathrm{Id} + U^j).$$

Convergence

Since $|P^j|_{s_j,r_j} \leq \varepsilon_j$, for any $\alpha \in K$, $F^j_\alpha = T_\alpha + P^j_\alpha \to T_\alpha$. Assuming $\psi^j = \mathrm{Id} - v^j \to \psi$ on K and $\Phi^j = \mathrm{Id} + U^j \to \Phi$ on $\mathbb{T}^n_{s/2} \times K$, then

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Let us set $\delta_1 = \sum_{j \geq 0} 8n\varepsilon_j / r_j$ and $\delta_2 = \sum_{j \geq 0} \varepsilon_j$, then

$$\begin{cases} \sum_{j \geq 0} \mathrm{Lip}(\tilde{v}_{j+1}) \leq \delta_1 \\ \sum_{j \geq 0} |\tilde{v}_{j+1}|_0 \leq \delta_2 \end{cases} \Longrightarrow \begin{cases} \sup_{j \geq 0} \mathrm{Lip}(v_{j+1}) \leq 2\delta_1 \leq 32n\varepsilon / r \leq 1/2 \\ \sum_{j \geq 0} |v_{j+1} - v_j|_0 \leq (1 + 2\delta_1)\delta_2 \leq 2\varepsilon. \end{cases}$$

Convergence

Since $|P^j|_{s_j,r_j} \leq \varepsilon_j$, for any $\alpha \in K$, $F^j_\alpha = T_\alpha + P^j_\alpha \to T_\alpha$. Assuming $\psi^j = \mathrm{Id} - v^j \to \psi$ on K and $\Phi^j = \mathrm{Id} + U^j \to \Phi$ on $\mathbb{T}^n_{s/2} \times K$, then

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With $\delta_3 = \sum_{j\geq 0} \sigma_j C(\sigma_j)\varepsilon_j$ and $\delta_4 = \sum_{j\geq 0} C(\sigma_j)\varepsilon_j$ we have

$$\begin{cases} \sum_{j\geq 0} |\tilde{U}^{j+1}_{\tilde{\psi}^j}|_{\hat{s}_j} \leq \delta_3 \\ \sum_{j\geq 0} |D\tilde{U}^{j+1}_{\tilde{\psi}^j}|_{\hat{s}_j} \leq \delta_4 \end{cases} \Longrightarrow \begin{cases} \sum_{j\geq 0} |U_{j+1} - U_j|_{s/2} \leq \delta_3 \leq 2\sigma_0 C(\sigma_0)\varepsilon \\ \sup_{j\geq 0} |DU_{j+1}|_{s/2} \leq 2\delta_4 \leq 1/2 \\ \sum_{j\geq 0} |D(U_{j+1} - U_j)|_{s/2} \leq 2\delta_4 \leq 4C(\sigma_0)\varepsilon. \end{cases}$$

THANK YOU