Stochastic properties of Lorentz gases

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Abstract

In these notes we review old and recent results on stochastic properties of periodic Lorentz gases with finite and infinite horizon, mainly described in terms of the displacement or flight function. For a short definition, one can think of a periodic Lorentz gas as a unit mass particle moving and bouncing elastically in a periodic grid of scatterers. We will focus on some results for the discrete time models (maps) that highlight the difference between finite and infinite horizon. In the last part, we also mention recent results for continuous time (flows).

Plan of the lecture notes. In Section 1, we provide a general description of discrete time (maps) Lorentz gases with finite and infinite horizon along with the required terminology, in particular for the displacement function and the flight function.

In Section 2, we present various stochastic properties of Lorentz gases, first for finite horizon in Subsection 2.1 and then for infinite horizon in Subsection 2.2. Every stochastic property in the finite horizon has an analogue in the infinite horizon, but with different types of scalling sequences. Among other properties, in this section, we will present the Local Limit Theorem for the the displacement function in the set up of the Lorentz gas with infinite horizon, as proved by Szász & Varjú in [45].

In Section 3, we will go over the main step of the Local Limit Theorem for the the displacement function in the set up of the Lorentz gas with infinite horizon [45], which requires among others the notions of Young towers (reviewed in Subsection 3.1.2) and perturbed transfer operators (reviewed in Subsection 3.2.2). In the same section we will also present the main steps for obtaining error terms in mixing for the infinite measure preserving Lorentz gas with infinite horizon as in [41].

In Section 4, we will present the Lorentz gas flow as a $\mathbb{Z}^d$ extension of the a suspension flow over the billiard map, referred to as Sinai billiard. We will state the mixing results for both finite and infinite horizon, explain why they are much harder to prove than for maps and review the main line of argument.

1 General description of periodic Lorentz gases (LG)

The Lorentz gas, a popular model of mathematical physics introduced by Lorentz in 1905 ([31]), is a dynamical system on the infinite billiard table obtained by removing strictly convex scatterers from either $\mathbb{R} \times \mathbb{S}$ or $\mathbb{R}^2$. This model describes the evolution of a point particle moving freely with unit velocity and elastic reflections off pairwise disjoint strictly convex obstacles (with smooth boundaries) located periodically in $\mathbb{Z}^d$: on the tube if $d = 1$ or the plane if $d = 2$.

For simplicity, we assume that the scatterers or obstacles are round disks. In all generality, the obstacles do not need to be round, as long as they are open convex sets with C^3 boundaries with nonzero curvatures and such that the obstacles have pairwise disjoint closures.

1.1 Tubular finite horizon LG

A Lorentz tube consists of a unit mass particle moving and bouncing elastically in a periodic grid of scatterers inside $\mathbb{R} \times \mathbb{S}^1$. Consider the following picture:

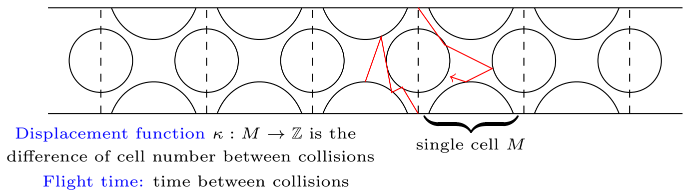


Figure 1: A tubular Lorentz map/flow (finite horizon)

By a periodic grid of scatterers, we mean that the obstacles (round scatterers) are placed periodically in tube (a cylinder). One can think of this tube as divided in cells, the dashed lines are only there to help portraying the division in cells. The dynamics is given by the motion of the particle: the particle reflects elastically against the obstacles and move along straight lines otherwise. We can distinguish between the dynamics on the cell, say M in the notation of the picture and the dynamics on the entire tube. One can look at the dynamics in discrete or continuous time. As we shall see, the continuous time dynamics is much harder to study.

A closer look at the discrete time dynamics on the cell. The dynamics on the cell $T : M \to M$ is called the Sinai billiard and it describes the dynamics from one collision to the next. The phase space is given by

$$M = \{(q, v) : q = \text{position at the boundary of the scatterers }, v = \text{speed}\}.$$

Recall that the particle goes at unit speed, so it is only the angle that matters. Denote the obstacles by O_j, with j in some index set. So, the point $x = (q, v) \in M$ can be described by $q \in \partial O_j$ and the angle $\varphi \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ with the normal vector to ∂O_j oriented outside O_j, as illustrated in Figure 2 below. So, we can be more precise and write

$$M = \{(q, v) : q = \text{position at the boundary of the scatterers }, v = \text{speed}\} = \bigcup_j \partial O_j \times \left[-\frac{\pi}{2}, \frac{\pi}{2}\right].$$

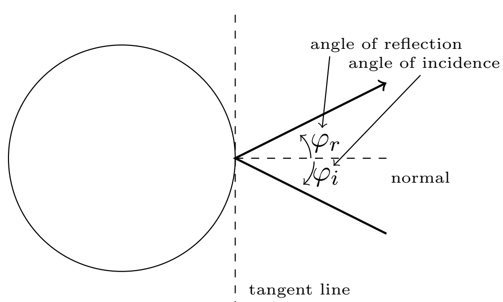


Figure 2: Elastic collision with a scatterer: $\varphi_i = \varphi_r = \varphi$.

The billiard map T preserves the measure $d\mu = \cos\varphi dq d\varphi$, Each cell is a finite measure preserving system (M, T, μ).

On every cell we have the same type of dynamics. This does not mean that in every cell, the trajectory is the same, but it follows the same evolution rule. For the Lorentz tube, we number the cells as $\ldots, -2, -1, 0, 1, 2 \ldots$ and thus, we can think of the displacement function as steps in a random walk (on $\mathbb{Z}$ in the picture above). The displacement function (through which we will record the stochastic properties) is $\kappa_0(x) = 0$ and for $n \geq 1$,

$$\kappa : M \to \mathbb{Z}, \quad \kappa_n(x) = \sum_{j=0}^{n-1} \kappa \circ T^j(x) : \text{ Cell number at time } n.$$

Entire discrete time dynamics: discrete time tubular LG. The entire dynamics on the tube can be seen as $\mathbb{Z}$ extension by κ of T. That is, define the entire dynamics along with its iterates by

$$\hat{M} = M \times Z, \quad \hat{T} : \hat{M} \to \hat{M}, \quad \hat{T}^n(x, \ell) = (Tx, \ell + \kappa_n(x)).$$

Clearly, $\hat{T}$ preserves the infinite measure $\hat{\mu} = \mu \times Leb_{\mathbb{Z}}$.

One thing to keep in mind about κ is that it has mean zero $\int \kappa d\mu = 0$, so it behaves in 'some sense' as a symmetric random walk. The reason for **mean zero of** κ is translational and reflectional symmetry. That is, rotating the frame by 180 degrees around any point in the one dimensional lattice preserves the configuration of scatterers, but it gives a minus sign to the displacement function κ. This implies that

$$\mu(\kappa = N) = \mu(\kappa = -N) \text{ which implies that } \int \kappa d\mu = 0.$$

Saying that the LG has **finite horizon** means that it is impossible to have a trajectory that does not intersect scatterers, and therefore κ is bounded.

1.2 Tubular infinite horizon LG

Everything that we said before about the tubular finite horizon stays the same in the infinite horizon with one crucial difference: in this case κ is unbounded, in fact $\kappa \notin L^2(\mu)$. In other words we no longer have 'some sort' of similarity with a random walk on $\mathbb{Z}$ with finite moments, but with a random walk on $\mathbb{Z}$ with infinite second moment.

The following picture illustrates infinite horizon:

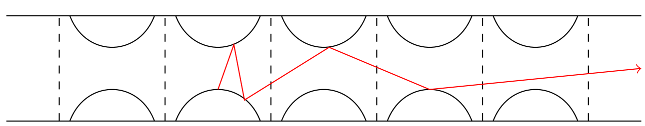


Figure 3: A tubular Lorentz map/flow (infinite horizon)

Contrary to the finite horizon, the particle can travel arbitrarily far along a corridor. The horizontal strip between the scatterers is a corridor.

1.3 An example of a two dimensional infinite horizon LG

Figure 4 illustrates an infinite horizon LG in dimension 2 with corridor directions: one horizontal, one vertical. There are different choices of dividing the lattice into cells, below is just one example: the red square in Figure 4 is just an example.

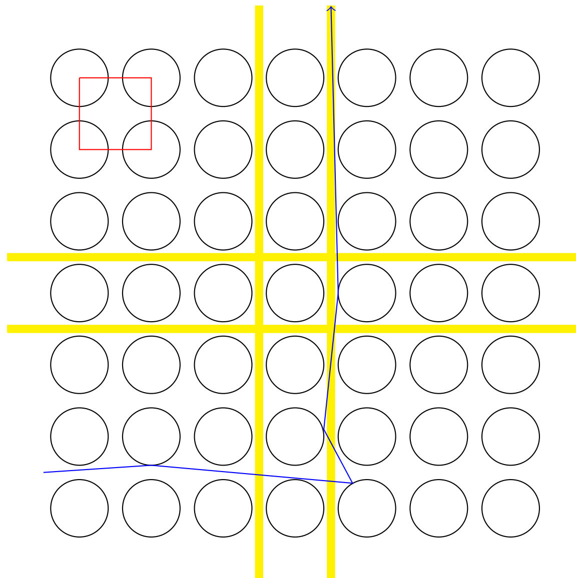


Figure 4: Lorentz gas with two corridor directions.

2 Stochastic properties of finite and infinite horizon discrete time LG

In this section, we present various stochastic properties of Lorentz gases, first for finite horizon in Subsection 2.1 and then for infinite horizon in Subsection 2.2. Every stochastic property in the finite horizon has an analogue in the infinite horizon, but with different types of scaling sequences.

We present stochastic properties in terms of $\kappa : M \to \mathbb{Z}^d$ but all the results mentioned below hold the flight function $V : M \to \mathbb{R}^d$, which gives the distance between consecutive collisions in $\mathbb{R}^d$. The reason for this is that V and κ are *cohomologous*:

$$V(x) = \kappa(x) + H(x) - H(Tx),$$

where $H - H \circ T$ is referred to as a bounded (mean zero) coboundary. To see this, note that for $H(x) = [x] - x$, one has $\kappa(x) = [T(x)] - [x]$ and

$$V(x) = T(x) - x = T(x) - [T(x)] + [T(x)] - [x] + [x] - x = -H(Tx) + H(x) + \kappa(x).$$

2.1 Finite horizon, $d = 1, 2$

In the finite horizon case, the central limit theorem (CLT)

$$\frac{\kappa_n}{\sqrt{n}} \to^d \mathcal{N}(0, \Sigma^2) \text{ where } \Sigma \text{ is a matrix with } \det \Sigma \neq 0 \tag{1}$$

was first proved by Bunimovich and Sinai in [8]. The statement above is for planar case, so $d = 2$. If $d = 1$, then $\Sigma = \sigma$ is a scalar.

The main steps of the proof in [8] are:

- Exponential bound on the *autocorrelations* of κ. In particular, they showed that are $C > 0$ and $\theta \in (0, 1)$ such that
$$\left| \int_M \kappa \cdot \kappa \circ T^n \, d\mu \right| \leq C\theta^n \text{ for all } n \geq 0.$$

This was established via Markov partitions and symbolic dynamics. This was the very hard step.

- Use the Green–Kubo formula

$$\Sigma^2 = \sum_{j=-\infty}^{\infty} \int_M \kappa \cdot \kappa \circ T^j \, d\mu$$

to express the asymptotic covariance matrix as the infinite sum of correlations. Using geometric arguments, it is shown in [8] that $\det \Sigma \neq 0$. (If $d = 1$, this means $\sigma^2 > 0$).

- Using a classical CLT criterion for weakly dependent stationary sequences, which essentially captures the summable decay of autocorrelation for κ , [8] obtained CLT.

In fact, in [8] the authors also proved the functional central limit theorem (also known as invariance principle). That is, for the planar LG with finite horizon, they showed that

$$\left(\frac{\kappa_{\lfloor nt \rfloor}}{\sqrt{n}} \right)_{t \in [0,1]} \xrightarrow{d} (W_t)_{t \in [0,1]}.$$

This expresses convergence in distribution in $C([0,1], \mathbb{R}^2)$ to a Brownian motion W_t with covariance matrix Σ (as above). This is a refinement of the CLT.

In the finite horizon case, Szász & Varjú in [44] established **local central limit theorem (LCLT)** for κ. Let $\Psi(x) = \frac{1}{(2\pi)^{d/2} \det(\Sigma)} e^{-\frac{1}{2} x \cdot \Sigma^{-2} x}$ be the density of the multivariate Gaussian.

Theorem 2.1 (LCLT for κ in the finite horizon as in [44])

$$n^{d/2} \, \mu(\kappa_n = N) \to_{n \to \infty} \Psi(N) \tag{2}$$

for any $N \in \mathbb{Z}$.

This is obtained via a Young tower construction [47] and Fourier transforms captured in terms perturbed transfer operators as in, for instance, [2]. (We will come back with an explanation in Subsection 3.2.1).

LCLT not only yields precise asymptotics for the displacement distribution, but also **implies recurrence for infinite measure LG**: the probability of returning to a fixed cell decays at a non-summable rate, ensuring that returns occur infinitely often with probability one.

This a consequence of the 'Schmidt–Conze' criterion for $\mathbb{Z}^d$ extensions over a probability preserving base: see Conze [13] and Schmidt [46]. In short, one recalls that conservativity/recurrence of $\hat{T}$ is equivalent to

$$\sum_n \hat{\mu} \left((M \times \{0\}) \cap \hat{T}^{-n} (M \times \{0\}) \right) = \sum_n \mu(\kappa_n = 0) = \infty.$$

To see the first equality, note that the quantity $\hat{\mu} \left((M \times \{0\}) \cap \hat{T}^{-n} (M \times \{0\}) \right)$ gives the probability of starting and returning after n collisions in the same cell, that is with displacement equal to 0; this quantity is therefore equal to $\mu(\kappa_n = 0)$.

We note that recurrence for the finite measure Sinai billiard map is automatic due the Poincaré recurrence theorem. For any set A with $\mu(A) > 0$, $\sum_{n \geq 1} 1_A(T^n x) = \infty$ for a.e. $x \in A$.

Mixing and mixing rates for the infinite measure $\hat{T}$ can also be obtained starting from LCLT. In short, one is interested in understanding the asymptotic behavior of $\int v \cdot w \circ \hat{T}^n$, for a suitable class of functions v, w defined on M.

Recall that for $v = w = 1_M$,

$$\hat{\mu}(M \cap \hat{T}^{-n} M) = \mu(\kappa_n = 0).$$

For a nice summary of this problem and how to generalize to different functions (other than 1_M) see [39].

In [38, 40], Pène later refined the LCLT in [44] in two respects:

- For *natural classes of observables* v_0, w_0 defined on M, Pène studied the asymptotics of

  $$\int_M v_0 1_{\{\kappa_n = N\}} \, w_0 \circ T^n \, d\mu \qquad \text{as } n \to \infty.$$

  This is referred to as 'Mixing Local Limit theorem'.
- *Speed in the LCLT.* In fact, Pène obtained a complete asymptotic expansion (to arbitrarily order) for

  $$\int_M v_0 1_{\{\kappa_n = N\}} \, w_0 \circ T^n \, d\mu, \quad N \in Z^d.$$

Using these two ingredients, among others, Pène [38] obtained the complete asymptotic expansion (to arbitrarily order) of

$$\int v \cdot w \circ T^n = \Psi(0) \, n^{-d/2} \int_{\hat{M}} v \, d\hat{\mu} \int_{\hat{M}} v \, d\hat{\mu} + \text{ complete expansion with precise constants}$$

for $\hat{M} = M \times Z^2$ and a natural class of functions $v, w \in L^1(\hat{\mu})$ referred to as dynamically Lipschitz functions. Here Ψ is as in (2).

Almost sure invariance principle (ASIP) and *law of iterated logarithms (LIL)* for finite horizon LG

Definition 2.2 *An $\mathbb{R}^d$ valued process $(Y_j)_{j \geq 0}$ defined on some probability space $(\Omega, \mathbb{P})$ satisfies ASIP if there exists a probability space $(\Omega_1, \mathbb{P}_1)$ (possibly larger than $(\Omega, \mathbb{P})$) and two processes $(Y_j^*)_{j \geq 0}$, $(Z_j)_{j \geq 0}$ defined on $(\Omega_1, \mathbb{P}_1)$ so that*

- *$(Y_0, Y_1, \ldots)$ is distributed as $(Y_0^*, Y_1^*, \ldots)$.*
- *$Z_0, Z_1, \ldots$ are independent and distributed as $\mathcal{N}(0, I)$.*
- *$\left|\sum_{j=0}^{n-1} Y_j^* - \Sigma \sum_{j=0}^{n-1} Z_j\right| = o(n^\varepsilon)$ for some $\varepsilon \in (0, 1/2)$ and for some non-degenerate covariance matrix Σ^2.*
  Equivalently, one says that $\left|\sum_{j=0}^{n-1} Y_j^* - \Sigma \cdot W_n\right| = o(n^\varepsilon)$, where $W_n = \sum_{j=0}^{n-1} Zj$ denotes standard Brownian motion on $\mathbb{R}^d$ at time n.

In short, $\sum_{j=0}^{n-1} Y_j$ satisfies an ASIP with rate n^ε for some $\varepsilon \in (0, \frac{1}{2})$ if there exists some non-degenerate covariance matrix Σ^2 such that, possibly on an enlarged probability space,

$$\left|\sum_{j=0}^{n-1} Y_j - \Sigma \cdot W_n\right| = o(n^\varepsilon) \tag{3}$$

almost surely as $n \to \infty$, where W_n the standard Brownian motion on $\mathbb{R}^d$ at time n.

It is known that the ASIP implies LIL:

$$\limsup_{n\to\infty} \frac{|\sum_{j=0}^{n-1} Y_j|}{\sqrt{n \cdot L(Ln)}} = a \qquad \text{almost surely, for some } a \in (0,\infty),$$

where $L(t) = \max(1, \log(t))$. This is because the LIL is known for the standard Brownian motion.

In [35, 36], Melbourne and Nicol proved that $(\kappa \circ T^j)_{j\geq 0}$ satisfies ASIP in the sense of Definition 2.2. In short,

$$|\kappa_n - \Sigma \cdot W_n| = o(n^{\varepsilon}), \mu \text{ almost surely, for 'good' } \varepsilon \in (0, 1/2). \tag{4}$$

Here Σ^2 is the covariance matrix in the CLT. The 'almost surely' w.r.t. the invariant measure μ (under T) was clarified by Korepanov [25]. An immediate consequence of this ASIP is LIL for κ, that is

$$\limsup_{n\to\infty} \frac{|\kappa_n|}{\sqrt{n \cdot L(Ln)}} = a \qquad \text{almost surely, for some } a \in (0,\infty).$$

One aim is to prove ASIP with rates, that is to obtain ASIP in (4) for ε very small. The rates in the ASIP were later improved by Gouëzel in [23] and also, later on by Korepanov [26].

2.2 Infinite horizon, $d = 1, 2$

Contrary to the case of finite horizon, in the infinite horizon case, κ fails to be $L^2(\mu)$. This was established in [45, Proposition 6].

Recall the collection of corridors of vectors depicted in the examples of Figures 3 and 4. By [45, Proposition 6], there exists a finite collection $Corr \subset \mathbb{Z}^d$ of vectors (the corridor vectors), such that, for any $\xi \in Corr$ there exists $C_\xi > 0$, as $N \to \infty$,

$$\mu(\kappa = N\xi) = C_\xi N^{-3}(1 + o(1)).$$

The event $\kappa = N\xi$ indicates the next collision with a scatterer at distance $|N||\xi|$ from the current position. This implies that $\mu(|\kappa| > N) = CN^{-2}(1 + o(1))$, for some $C > 0$; so κ barely fails to be $L^2(\mu)$. For generalizations of this statement with expansions in terms of N along with the more than one type of obstacles being present, see [41, Lemma 4.2] or accounting for the precise size of the scatterers, see [4, Appendix A].

This tail behaviour determines a different type of scaling in the CLT. In this case we speak of CLT with a non-standard normalizing sequence, under suitable non-degeneracy conditions:

$$\frac{\kappa_n}{\sqrt{n \log n}} \to^d \mathcal{N}(0, \Sigma^2) \tag{5}$$

where $\det \Sigma \neq 0$ and Σ is obtained by summation over corridors. The non-degeneracy conditions, relevant only if the scatterers are not positioned at lattice points, are as follows: a) in the case $d = 2$, one requires that there exist at least two non-parallel collisionless trajectories in the interior of the periodic domain; b) when $d = 1$, one requires that there exists a collisionless trajectory not orthogonal to the direction of the $\mathbb{Z}$-cover or extension, which is equivalent to our assumption that κ is unbounded.

We note upfront that the Infinite horizon is not special just because the variance is infinite; it is special because we have $\alpha = 2$ stable law, where sums are not outlier-dominated (as in $\alpha < 2$) but built from the cumulative effect of many moderate extremes, giving the Gaussian limit with $\sqrt{n \log n}$ scaling.

The form of the CLT in (5) was first conjectured by Bleher [9] and proved rigorously via two different methods establishing stronger versions of this type of CLT:

(I) Szász and Varjú [45] proved a Local Limit Theorem (LLT), as stated below.

Let Ψ is the density of the Gaussian in the non-standard CLT (5). (In fact, one can describe Ψ completely independent of the CLT (5), just writing down the formula $\Psi(x) = \frac{1}{(2\pi)^{d/2} \det(\Sigma)} e^{-\frac{1}{2}x \cdot \Sigma^2 x}$).

Theorem 2.3 (LLT for κ as in [45])

$$(n \log n)^{d/2} \mu(\kappa_n = N) \to_{n \to \infty} \Psi(N), \text{ for any } N \in \mathbb{Z}.$$

The method consists in exploiting the associated Young tower, a double or conditional probability (to be recalled shortly) and some abstract results by Bálint and Gouëzel [6]. In Section 3, we will go over the terminology and describe the main steps of the proof of Theorem 2.3.

(II) Chernov and Dolgopyat [12] established a functional limit theorem (weak invariance principle), namely

For $s \in (0,1)$, $\frac{\kappa_{\lfloor ns \rfloor} + \{ns\}(\kappa_{\lfloor ns \rfloor + 1} - \kappa_{\lfloor ns \rfloor})}{\sqrt{n \log n}}$ converges as $n \to \infty$
to a Brownian motion with mean 0 and covariance matrix Σ^2.

The method of proof in [12] exploits exponential mixing for the sequence $\{\kappa \circ T^n\}_{n \geq 1}$. The authors develop an argument based on standard pairs to establish a bound on the correlations for κ:

There exist $\vartheta \in (0,1)$ and $C > 0$so that
$\left| \int_M \kappa \cdot \kappa \circ T^n \, d\mu \right| \leq C \cdot \vartheta^n$ for all $n \geq 1$.

Similar to the finite horizon case, LLT implies **recurrence** and it can be used (in fact, some techniques that are used in the proof together with many other ingeredients) to establish:

Mixing and mixing rates for a natural class of functions. This was done by Pène and Terhesiu in [41] by obtaining error rates in the LLT. In very rough terms, a typical mixing with rates in [41] says that for 'good' functions (not necessarily compactly supported) $v, w : \hat{M} = M \times \mathbb{Z} \to \mathbb{R}$,

$$\int_M v \cdot w \circ \hat{T}^n \, d\hat{\mu} = \frac{\Psi(0)}{(n \log n)^{d/2}} \int_M v \, d\hat{\mu} \int_M w \, d\hat{\mu} + O\left(\frac{1}{(n \log n)^{d/2} \log n} \right).$$

For non-zero mean function the error term is optimal: it cannot be improved for $v = w = 1_M$. If v, w have mean zero, much better error terms are obtained in [41, Theorem 2.5]. In subsection 3.2.2 we will review the main steps of the proof.

Generalized *law of iterated logarithms (LIL)* for infinite horizon LG

Recall that in the finite horizon, κ satisfies the classical ASIP as in Definition 2.2.

In [7], Bálint and Terhesiu obtained a version of ASIP and generalized LIL for the infinite horizon. We recall briefly this result. Recall $\mu(\kappa = N) = CN^{-3}(1 + o(1))$ and define

$$c_n = \sqrt{2C \cdot n \cdot L(n) \cdot LL(n) \cdot (1 + LLn \sin^2(LLLn))} \text{ for } L(t) = \max(1, \log t).$$

A main feature of the sequence c_n is that $\sum_n \mu(|\kappa| > c_n) < \infty$ so that by truncating at c_n, one can work with a new random variable $\kappa 1_{|\kappa|<c_n}$ that is $L^2(\mu)$. To some extent borrow ideas from variables with second moments. This is only a small part of the proof, but at least it gives an intuition of why truncating at a sequence, such as c_n, is helpful.

A version of ASIP. In [7], the authors show that there exist a probability space $(\Omega^*, \mathcal{F}^*, \mathbb{P}^*)$ and two sequences of random vectors $(v_j^*)_{j\geq 0}$, $(Z_j)_{j\geq 0}$ so that

- $(v_0^*, v_1^*, \ldots)$ is distributed as $(\kappa, \kappa \circ T, \ldots)$;
- the vectors $Z_j, j \geq 0$ are independent and distributed as $\mathcal{N}(0, I)$;

such that, almost surely, as $n \to \infty$,

$$\left| \sum_{j=0}^{n-1} v_j^* - \Gamma_n \sum_{j=0}^{n-1} Z_j \right| = o(c_n),$$

where Γ_n is the covariance matrix given by $\Gamma_n^2 = Cov(\kappa 1_{|\kappa|<c_n}) = L(c_n)\Sigma^2$ with Σ as in (5). Here, $\sum_{j=0}^{n-1} Z_j = W_n$, where W_n is the standard Brownian motion in dimension d, at time n. For this version of ASIP, the information about the *small tail* $\mu(\kappa = N) = CN^{-3}(1 + o(1))$ is not required, all that is needed is that κ is regularly varying of index -2. If $\kappa : M \to \mathbb{Z}$ this means that $\mu(|\kappa| > N) = CN^{-2}(1 + o(1))$.

In short, this is saying that on an enlarged probability space,

$$\left| \kappa_n - \Gamma_n \sum_{j=0}^{n-1} Z_j \right| = o(c_n).$$

This is an analogue of Einmahl's result [21]) obtained in an independent scenario (so the sequence of independent random variables $\kappa \circ T^j$ is replaced by a sequence of dependent random variables X_j). The proof in [7] adapts the strategy of [23] to case of non-$L^2(\mu)$ observables after truncating at different levels (with truncation at c_n included). The choice of the truncation level is delicate and the method requires very precise bounds on the second and the fourth moment for partial sums of the appropriately truncated random variables. To control these moments the authors use the standard pair method of [12], improving the estimates in [12, 5].

A serious difference between finite horizon ASIP $|\kappa_n - \Sigma \cdot W_n| = o(n^\varepsilon)$, $\varepsilon \in (0, 1/2)$ as mentioned in (4) and the version of ASIP in the infinite horizon $\left|\kappa_n - \Gamma_n \sum_{j=0}^{n-1} Z_j\right| = o(c_n)$ is that

- Finite horizon. Recall the scaling sequence in the CLT is $\sqrt{n}$: see equation (1). So, dividing by $\sqrt{n}$ in $|\kappa_n - \Sigma \cdot W_n| = o(n^\varepsilon)$ gives $o(n^\varepsilon)$.
- Infinite horizon. Recall the scaling sequence in the CLT is $\sqrt{n \log n}$: see equation (5). In this case, we see that $\frac{c_n}{\sqrt{n \log n}} \to \infty$.

  When compared to the standard ASIP, a specific feature of this ASIP version is that the rescaling of Brownian motion with the truncated covariance matrix Γ_n , occurs only for variables in the non-standard domain of attraction of the normal law. In the previous scenario, there is no need for truncation.

Is this c_n optimal, and as such is is this ASIP version optimal? It seems plausible if one takes a closer look at Einmahl's result (see [21, 22] and references therein), but we are not aware of a rigorous proof.

Generalized law of the iterated logarithm (LIL)

The main purpose of the ASIP version in the IID random variables with the same tail as κ (see [21, 22]) is the implication of a precise form of a generalized LIL

Exploiting a consequence of the above ASIP, using the same argument as in [22],

$$\limsup_{n\to\infty} \frac{|\kappa_n|}{c_n} = 1$$

3 Main ingredients used in the proof of LLT in the infinite horizon by Szász and Varjú [45]

The purpose of this section is to recall the main ingredients and steps in the proof of LLT, which we now recall.

Theorem 3.1 (LLT for κ as in [45])

$$(n\log n)^{d/2}\,\mu(\kappa_n = N) \to_{n\to\infty} \Psi(N), \text{ for any } N \in \mathbb{Z}.$$

While reviewing the main steps, we will also review the additional steps in obtaining error term in LLT and mixing for LG as in [41].

Two main ingredients used in [45] are

- Young towers for Sinai billiards as put forward by Young [47] and Chernov [10].
- Transfer operators on the tower (and base of the tower) along with their twisted or perturbed version.

3.1 Young towers

The purpose of this subsection is to get familiar with the notion of the tower in the easier setting of a one dimensional maps as in Subsection 3.1.1 and understand the complications that arise in the Sinai billiard setting as in Subsection 3.1.2. In short, we aim to understand the terminology of the following result.

Theorem 3.2 (Young [47] and Chernov [10]) *The Sinai billiard map can be modelled via a Young tower.*

3.1.1 Young towers in a much easier setting

As to get across the idea of the Young tower we start with a much easier setting, namely that of the map considered by Liverani *et al.* in [29]:

$$f(x) = \begin{cases} x(1+2^\alpha x^\alpha), & 0 < x < \frac{1}{2} \\ 2x-1, & \frac{1}{2} < x < 1 \end{cases}.$$

For $\alpha \in (0,1)$, f preserves a probability measure μ that is absolutely continuous w.r.t. Lebesgue measure.

The tower for f is constructed in [48]. This map exhibits intermittent behavior due to a neutral fixed point at $x = 0$.

Although the original map f has neutral behavior near 0 ($f'(0) = 1$), one can accelerate f (aka induce away from the problematic region 0 as to obtain a new map, a uniformly expanding map.

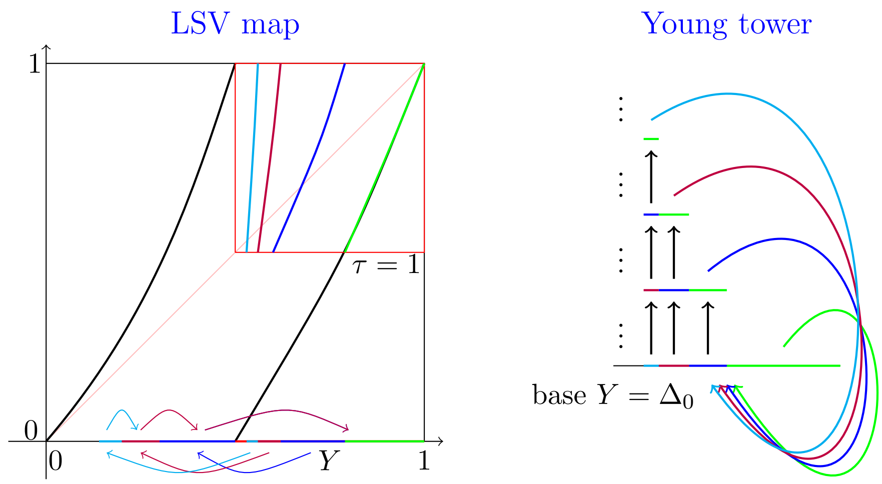


Figure 5: The LSV map and first return to $Y = [\frac{1}{2}, 1]$, and the corresponding Young tower

Return time function. We define the first return time to Y by

$$\tau(x) := \min\{n \geq 1 \mid f^n(x) \in Y\}.$$

Induced map and uniform expansion. The *induced map* (or first return map)

$$f_Y : Y \to Y, \qquad f_Y(x) = f^{\tau(x)}(x),$$

is uniformly expanding on each level set

$$Y_j = \{x \in Y : \tau(x) = j\}.$$

The induced map f_Y is uniformly expanding and preserves a measure $\mu_Y \ll Leb$. The map f_Y is *Gibbs–Markov* roughly, infinite branch uniformly expanding maps with bounded distortion and big images. Compared to Young towers, limit theorems are much easier to obtain for Gibbs-Markov maps than for Young towers. In Subsection 3.2.1 we review the more precise definition of Gibbs-Markov maps.

Partition of the base. We partition Y into level sets of the return time:

$$Y_j := \{x \in Y \mid \tau(x) = j\}, \qquad j \in \mathbb{N}.$$

Tower. Define the tower space

$$\Delta := \{(x, \ell) \in Y \times \mathbb{Z}_{\geq 0} \mid 0 \leq \ell < \tau(x)\} = \bigsqcup_{j \geq 1} \bigsqcup_{\ell=0}^{j-1} (Y_j \times \{\ell\}).$$

Tower map. The tower map $F : \Delta \to \Delta$ is given by

$$F(x, \ell) = \begin{cases} (x, \ell + 1), & \text{if } \ell + 1 < \tau(x), \\ (f^{\tau(x)}(x), 0), & \text{if } \ell + 1 = \tau(x). \end{cases}$$

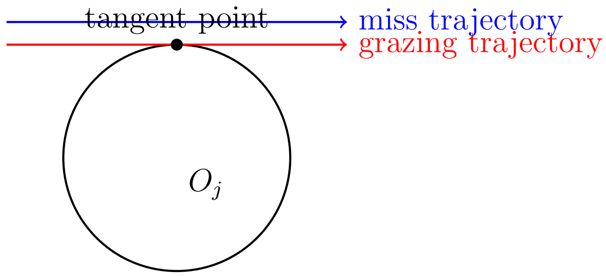


Figure 6: Discontinuities in the billiard map due to grazing.

Thus, under iteration of F, a point moves vertically up the tower until reaching the top, and then jumps to the base according to the return dynamics.

The tower map F commutes with f:

$$\pi : \Delta \to [0,1], \quad \pi(x,\ell) = f^\ell(x), \quad f \circ \pi = \pi \circ F.$$

3.1.2 Main steps for constructing a Young tower of the Sinai billiard

Compared to the previous 'easy' example, the Young tower construction for the Sinai billiard is much more complicated due to the following reasons:

1. The base is two dimensional with a contracting and an expanding direction. More importantly, returns to the partition elements are not onto the whole base, but onto entire strips in the unstable direction (as it is usually the case with hyperbolic maps).
2. The billiard map has discontinuities, also called singularities, and they are caused by grazing collision.

   Grazing collision phenomena at $\varphi = \pm\frac{\pi}{2}$: there exist nearby trajectories with no collisions. In other words, there exist parallel lines to the trajectory that just miss hitting a scatterer, see Figure 6.
3. The height of the tower τ is not a first return time, but a 'good' return time to a subset of the phase space M. But, for the tower map, τ is the first return time to the base.

Hyperbolicity A map $T : M \to M$ is *hyperbolic* if there is a continuous, DT-invariant splitting of the tangent bundle TM of the phase space M, into stable and unstable subspaces: $TM = \bigcup_x E^s_x \oplus E^u_x$, and there are $\lambda \in (0,1)$ and $C, \alpha > 0$ such that

- $D_xT : E^s_x \to E^s_{Tx}$ is onto for all $x \in M$ and

$$\|D_xT^n v\| \le C\lambda^n \text{ for all } v \in E^s_x \text{ and } n \in \mathbb{N}.$$

- $D_xT^{-1} : E^u_x \to E^u_{T^{-1}x}$ is onto for all $x \in M$ and

$$\|D_xT^{-n} v\| \le C\lambda^n \text{ for all } v \in E^u_x \text{ and } n \in \mathbb{N}.$$

- The subspaces E^s_x and E^u_x vary continuously in x, and have mutual angles $\angle(E^s_x, E^u_x) \ge \alpha$.

Here DT is the Jacobian matrix, which is the higher dimensional generalization of the derivative.

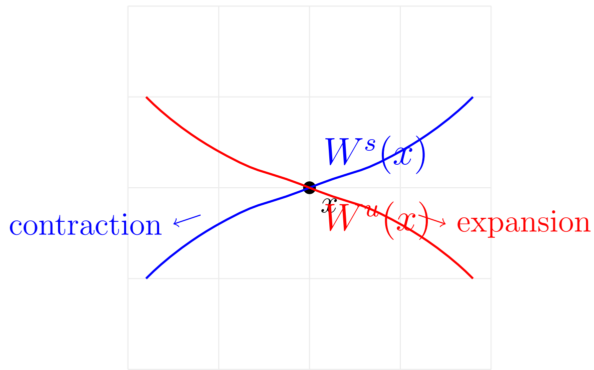


Figure 7: The stable and unstable manifolds at intersect transversally at x.

Hyperbolicity implies the existence, at least locally, of stable and unstable manifolds (curves) in the space M. The **stable** curve at x is

$$W^s(x) = \{y \in M : d(T^n x, T^n y)_{n \to \infty} \to 0\}.$$

The **unstable** curve at x is

$$W^u(x) = \{y \in M : d(T^{-n} x, T^{-n} y)_{n \to \infty} \to 0\}.$$

$W^s(x), W^u(x)$ give the contraction, respectively expansion (and in our setting, the contraction/expansion is exponential) and they intersect transversally at x, see Figure 7.

The tangent lines to $W^s(x), W^u(x)$ at x in the tangent spaces $T_x M$ are $E^s(x), E^u(x)$.

Set of singularities, grazing collisions Recall the Sinai billiard map $T : M \to M$, where

$$M = \{(q, v) : q = \text{position at the boundary of the scatterers}, v = \text{speed}\} = \cup_j \partial O_j \times \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

Let $S_0 = \{\varphi = \pm\frac{\pi}{2}\}$ be the set of grazing collisions of T. Iteration of the dynamics gives

$$S_n = \begin{cases} \cup_{i=0}^{n} T^{-i}(S_0), & n \geq 0 \\ \cup_{i=0}^{-n} T^{i}(S_0), & n < 0. \end{cases}.$$

Since T^n is not properly defined on S_n, E^s_x and $W^s(x)$ are not properly defined for $x \in \cup_{n \geq 0} S_n$. Also, $W^s(x)$ cannot intersect S_0, which suggests that

Even if $W^s(x)$ is defined, it can be arbitrarily short.

This further suggests that work is required (and this is part of Pesin theory [43]) to ensure that

$$length(W^s), length(W^u(T^n x)) > 0.$$

It suffices to show that $T^n(x)$ approaches S_0 at a slower rate than the contraction in the stable direction and $T^{-n}(x)$ approaches S_0 at a slower rate than the expansion in the unstable direction. To see the reason for this, start with a stable leaf $W^s(x)$ of fixed length. If you iterate this stable leaf forward, it gets shorter exponentially fast:

$$length(W^s(T^n)) \leq C\, \lambda^n \; length(W^s(x)).$$

If $T^n(x)$ is close to S_0 but not **that** exponentially close (i.e. larger than λ^n), then

$$length(W^s(T^n)) < d(T^n(x)\,,\,S_0)$$

and it will not be cut. If this holds for all $n \geq 1$, $W^s(x)$ will never be cut.

Figure 8 below gives an explanation of singularity lines in the phase space in terms of grazing collisions with some scatterers.

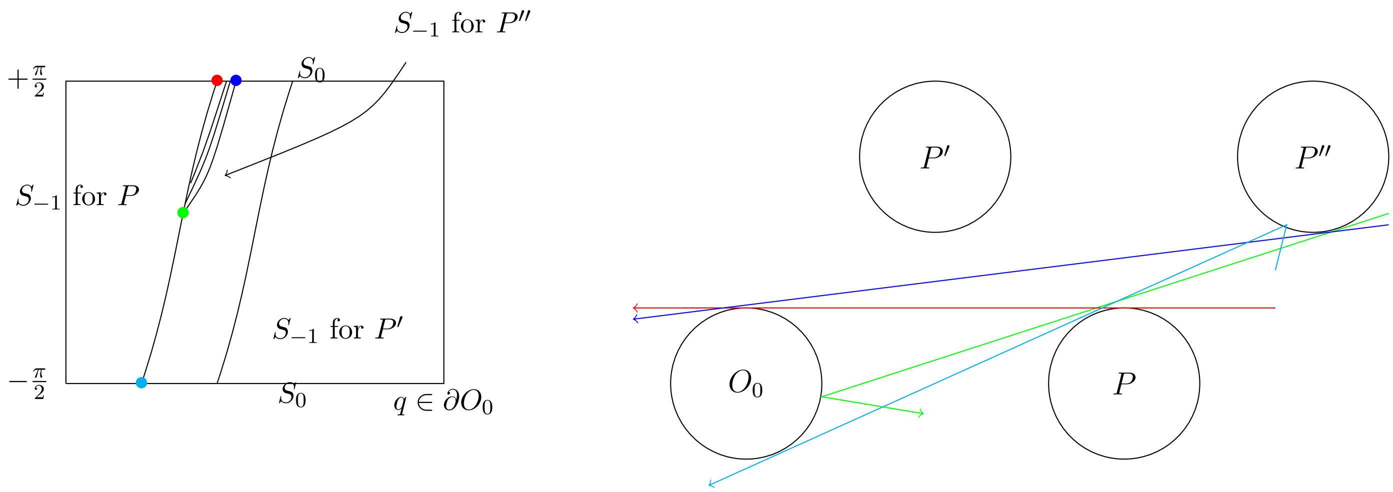


Figure 8: Points on S_{-1} refer to collisions on some scatterer, following a grazing collision at the previous scatterer. The different curves S_{-1} correspond to different scatterers at this previous grazing collision. Intersection points of $S_0 \cap S_{-1}$ or of different pieces of S_{-1} refer to double grazing collisions.

Some details on the steps of the YT construction. For uniformly hyperbolic billiards with singularities, such as Sinai billiard, one can try to find an induced map $F = T^R$ so that

1. it is uniformly hyperbolic
2. has long stable and unstable curves for every point in its domain.

   Roughly, we say that points x in the base are 'good' if

   - have long stable curves $W^s(x)$
   - when they return to the base, they have long unstable curves $W^u(T^R(x))$
3. has good distortion properties, that is good control of $\frac{DF^n(x)}{DF^n(y)}$, x, y in the same partition (at time n) element.

Item 2 above is really tricky, because a dense set of points in M have arbitrarily short stable and/or unstable manifolds. Their forward/backward orbit comes 'too close and too soon' to S_0.

Young [48] and Chernov [10] induced to a set of points $x \in M$ with long stable/unstable curves that contain a Cantor set Δ_0 of positive measure. We note that unlike in the previous $1D$ example, the return time is not necessarily a first return. An impression of the base of a Young tower is depicted in Figure 9 below.

Description of the Young tower

- The base of the tower is Δ_0 and it is partitioned into at most countably many subsets $\Delta_{0,i}$, $i \in \mathbb{N}$.

  On each partition element $\Delta_{0,i}$, the return function R is constant: $R|_{\Delta_{0,i}} = R_i$.

  Each $\Delta_{0,i}$ maps hyperbolically into the base after an number R_i iterates of the billiard map T.

- One constructs the tower map T_Δ is defined on whole tower $\Delta = \bigsqcup_i \bigsqcup_{j=0}^{R_i - 1} \Delta_{\ell,i}$ (where, for fixed i, $\Delta_{\ell,i}$ are copies of each other for $0 \le \ell < R_i$) as

$$T_\Delta(x, \ell) = \begin{cases} x \in \Delta_{\ell+1,i} & \text{if } 0 \le \ell < R_i - 1; \\ T^{R_i}(x) \in \Delta_0 & \text{if } \ell = R_i - 1; \end{cases}$$

- The projection $\pi : \Delta \to M$, $\pi(\Delta) = M \bmod \mu$, defined by $\pi(x \in \Delta_{j,i}) = T^j(x)$, satisfies

$$\pi \circ T_\Delta = T \circ \pi.$$

- The return map $T_{\Delta_0} = T^r : \Delta_0 \to \Delta_0$ where $r(x) = r_i$ if $x \in \Delta_{0,i}$, is countably piecewise hyperbolic, it preserves a measure ν_0 that is absolutely continuous w.r.t. Liouville measure μ.

  On the base of the tower, the **first return time** $\sigma : \Delta_0 \to \mathbb{N}$ satisfies $\sigma(x) = r(\pi(x))$ for $x \in \Delta_0$. On the tower Δ, one speaks of the **first return map** $F : \Delta_0 \to \Delta_0$, $F(x) = T_\Delta^{\sigma(x)}(x)$.

- By lifting ν_0 up the tower Δ, one obtains a T_Δ-invariant measure:

$$\mu_\Delta(E) = \sum_n \nu_0(F^{-n}E \cap \{r > n\}).$$

Why a fat Cantor set? The details of the construction are too elaborate to discuss. They are the hard core of the works [47] and [10]. But let us mention why it is a tick Cantor set.

The first thought one might have is that the base of the tower is a rectangle (previously, in the easy example, we had an interval). But this cannot happen because one has to exclude the 'non good' points. Recall that points x in the base are 'good' if

- have long stable curves $W^s(x)$
- when they return to the base, they have long unstable curves $W^u(T^R(x))$

The non good points form a dense set because they contain all the singularity lines $S_{\pm n}$. One needs to exclude points that come too early and too close to $S_{\pm n}$.

S_{+1} and S_{-1} have neighborhoods that have to be removed. The same holds for S_{+2} and S_{-2}, but these neighborhoods are already (proportionally) thinner than those for S_{+1} and S_{-1}. This continues the same way for all $n > 2$. By analogy with the middle third Cantor set, one sees that we the base has to be a Cantor set: see figure 9 and caption.

Why positive Lebesgue measure? Just recall the construction of a fat Cantor set: see figure 10.

The Cantor set with positive Lebesgue measure is necessary. If it had zero measure then $\pi(\Delta)$ cannot be equal to M mod μ, which is part of the construction of the Young tower.

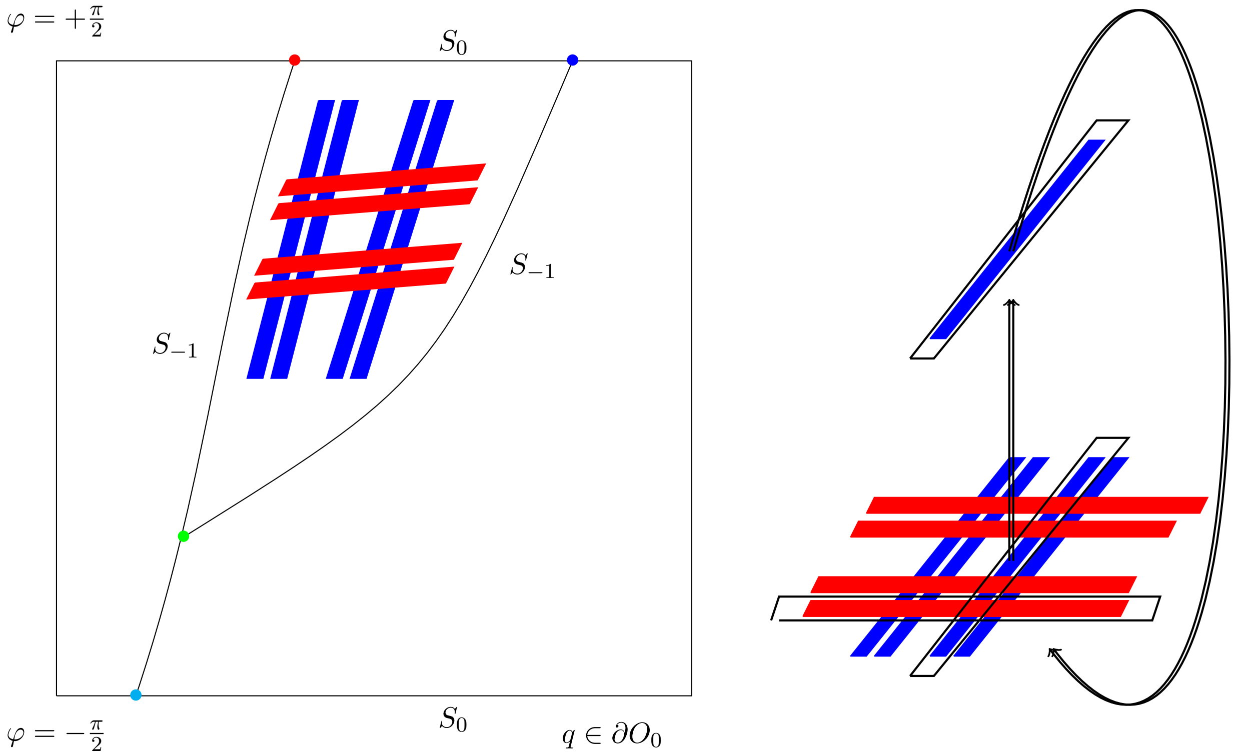


Figure 9: The base of the Young tower is a **thick** Cantor set (i.e., of positive Lebesgue measure) obtained as the intersection of a thick Cantor sets of stable leaves (blue) and a a thick Cantor sets of unstable leaves (red) inside the phase space. In the Young tower, strips in the stable direction form the partition $\Delta_{0,i}$ of the base. When returning to the base, they map to strips in the unstable direction.

Reducing to a one-dimensional Young tower. Using that T_Δ is hyperbolic one can reduce to a a one dimensional Young tower by factorizing/collapsing the stable curves to points inside unstable curves.

To avoid extra notation, we will still call it the **one-dimensional tower** $(\Delta, T_\Delta, \mu_\Delta)$. The base map, that is the first return map of the of this one dimensional tower map, $(\Delta_0, F = T_\Delta^\sigma, \nu_0)$ is Gibbs-Markov.

The one-dimensional tower map T_Δ can be made uniformly expanding after adjusting the metric. Using

- exponential returns: $\nu_0(r > n) \leq Ce^{-na}$ for some $C, a > 0$,
- hyperbolicity of $T_{\Delta_0} : \Delta_0 \to \Delta_0$,

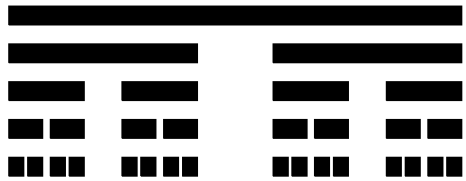
one can a $\lambda > 1$ so that for the metric

$$d_\Delta(x,y) = \begin{cases} \lambda^\ell d(x,y) & \text{if } x, y \in \Delta_{\ell,i} \text{ for some } \ell, i; \\ \infty & \text{else} \end{cases}$$

T_Δ is uniformly expanding. Here $d(x,y)$ is the Euclidean distance.

3.2 Role of transfer operators in the proof of LLT in [45]

While we recall the main steps in the proof of LLT in [45] using Fourier transforms or (perturbed transfer operators), we will use the more detailed framework in [41] which also gives error terms in LLT and mixing for the infinite horizon LG.



$p_1 = 1/6$
$p_2 = 1/5$
$p_3 = 1/10$
$p_3 = 1/9$

Figure 10: The Cantor set is fat if the sum of proportions of holes $\sum_k p_k < \infty$

3.2.1 Transfer operators for Gibbs-Markov maps and non-standard CLT and LLT

To get the idea of the role of the transfer operator, let us first look at the easier case of Gibbs-Markov maps. For Gibbs-Markov maps, Aaronson and Denker [3] obtained a local limit theorem for observables in the domain of attraction of a normal distribution that are not L^2 wrt the invariant measure. This type of observables have a behaviour that are similar to the displacement κ. The analysis comes down to working with a perturbed version of the transfer operator. In the literature, this method of using transfer operators for obtaining limit theorems (or local limit theorems) is referred to as **'Nagaev-Guivarch-Aaronson and Denker' method**, see [24].

To simplify even further let us review

• **I.I.D. set up** Consider the sequence of i.i.d. random variables $(X_j)_{j\geq 0}$ on $(\Omega, \mathcal{F}, \mathbb{P})$, $X_j : \Omega \to \mathbb{Z}^d$, $d = 1, 2$ with $\mathbb{E}(X_j) = 0$ and assume that

$$\mathbb{P}(X_j = wN) = \frac{c_w}{N^3} + O\left(\frac{1}{N^4}\right) \text{ as } N \to \infty, \text{ for all } w \in \mathbb{Z}^d \text{ and for some } c_w > 0. \qquad (6)$$

Let us write $w \in \mathbb{Z}^d$ as column vector. Assume further that $\sum_{w\in\mathbb{Z}^d} c_w w \cdot w^T = A$ for some positive symmetric matrix A, so $A_{ij} = \sum_{w\in\mathbb{Z}^d} c_w w_i w_j$.

This is the i.i.d. set up in which X_j plays the role of $\kappa \circ T^j$. The second term is necessary in order to obtain LLT with error terms.

The *Fourier transform* or *characteristic function* of X_j is

$$\psi(t) = \mathbb{E}(e^{it\cdot X_j}) \text{ for the } X_j \text{ with } t \in \mathbb{R}^d.$$

where $t \in \mathbb{R}^d$ and X_j both written as column vectors, and $t \cdot X_j$ denotes the scalar product of these vectors. A calculation using (6) gives the behaviour of $\psi(t)$ as $t \to 0$:

$$\begin{aligned}
1-\psi(t) &= 1 + it\cdot\mathbb{E}(X_j) - \sum_{w\in\mathbb{Z}^d}\sum_{N=1}^{\infty} e^{it\cdot wN}\mathbb{P}(X_j = wN) \\
&= \sum_{w\in\mathbb{Z}^d}\sum_{N<1/|t|}(1+it\cdot wN - e^{it\cdot wN})\mathbb{P}(X_j = wN) + O(t^2) \\
&= \sum_{w\in\mathbb{Z}^d}\sum_{N<1/|t|}(1+it\cdot wN - e^{it\cdot wN})\left(\frac{c_w}{N^3} + O\left(\frac{1}{N^4}\right)\right) + O(t^2) \\
&= \log(1/|t|)\left(\Big(\sum_{w\in\mathbb{Z}^d} c_w w\cdot w^T\Big)t\right)\cdot t + O(t^2) = \log(1/|t|)\, At\cdot t + O(t^2).
\end{aligned}$$

where $t^2 = t\cdot t = \|t\|^2 = \sum_{k=1}^d t_k^2$. In short, as $t\to 0$,

$$1-\psi(t) = \log(1/|t|)\, At\cdot t + O(t^2) \text{ and thus, } \psi(t) = \exp\left(-\frac{\log(1/|t|)\, At\cdot t}{2} + O(t^2)\right). \tag{7}$$

Write $S_n = \sum_{j=0}^{n-1} X_j$. Due to independence,

$$\mathbb{E}(e^{it\cdot S_n}) = \psi(t)^n = \exp\left(n\left(-\frac{At^2\log(1/|t|}{2} + O(t^2)\right)\right) \text{ as } t\to 0.$$

• **Non-standard CLT.** Replacing t with $\frac{t}{\sqrt{n\log n}} \to 0$, as $n\to\infty$,

$$\mathbb{E}(e^{it\cdot\frac{S_n}{\sqrt{n\log n}}}) \to e^{-\frac{At\cdot t}{2}} \text{ as } n\to\infty, \text{ for any } t\in\mathbb{R}^d.$$

Since $e^{-\frac{At^2}{2}}$ is the characteristic function of a Gaussian random variable $\mathcal{N}(0,A)$, the Levy continuity theorem implies that

$$\frac{S_n}{\sqrt{n\log n}} \to^d \mathcal{N}(0,A) \text{ as } n\to\infty. \tag{8}$$

• **LLT.** Let $N\in\mathbb{Z}^d$ be a column vector. Start with the Fourier inversion formula

$$\mathbb{P}(S_n = N) = \frac{1}{(2\pi)^d}\int_{[-\pi,\pi]^d} e^{-it\cdot N}\psi(t)^n\, dt.$$

(In fact, looking at $\psi(t)^n$ as a Fourier series, one can think of $\mathbb{P}(S_n = N)$ as the N-th Fourier coefficient)

One property of $\psi(t)^n$ in the i.i.d. set up is that for $t\in\mathbb{R}^2$, **outside a neighborhood** of $0\in\mathbb{R}^d$,

$$|\psi(t)^n| \le \varepsilon_0^n, \text{ for some } \varepsilon_0\in(0,1).$$

Thus, there exists $\delta > 0$ so that

$$\mathbb{P}(S_n = N) = \frac{1}{(2\pi)^d}\int_{[-\delta,\delta]^d} e^{-it\cdot N}\psi(t)^n\, dt + O(\varepsilon_0^n).$$

A change of variables $t \to \frac{t}{\sqrt{n \log n}}$ together with some elementary but not necessarily short calculations (see the chain of equations around [41, Formulas (98) and (99)]) gives that for any $N \in \mathbb{Z}^d$,

$$\mathbb{P}(S_n = N) = \frac{\Phi(N)}{(n \log n)^{d/2}} + O\left(\frac{1}{(n \log n)^{d/2} \log n}\right), \tag{9}$$

where Φ is the density of the random variable distributed according to $\mathcal{N}(0, A)$.

The **message** of the i.i.d. set up is that once you have the asymptotics of the characteristic function (7), non-standard CLT (8) and LLT (with rates) (9) are typical consequences.

In a non-independent set up, one has to 'work hard' to get something similar to (7). 'How hard' depends on the set up. As we shall see below, for Gibbs-Markov maps the work is not very hard. For Sinai billiards, we have a completely different story.

Gibbs-Markov maps along with transfer operator on a suitable Banach space. Let (Y, μ_Y) be a probability space, with a non-trivial countable partition $\{a\}$. Let $T_Y : Y \to Y$ be a topologically mixing ergodic measure-preserving transformation, piecewise continuous w.r.t. $\{a\}$. Define the *separation time* $s(y, y')$ to be the least integer $n \geq 0$ such that $T^n y$ and $T^n y'$ lie in distinct partition elements. Assuming that $s(y, y') = \infty$ if and only if $y = y'$ one obtains that $d_\theta(y, y') = \theta^{s(y,y')}$ for $\theta \in (0, 1)$ is a metric.

Let $\varphi = \frac{d\mu_Y}{d\mu_Y \circ T_Y} : Y \to \mathbb{R}$. We say that T is a *Gibbs-Markov map* if the following hold w.r.t. the countable partition $\{a\}$:

- $T_Y|_a : a \to T(a)$ is a measurable bijection for each a such that $T(a)$ is the union of elements $\{a\}$ modμ_Y;
- $\inf_a \mu_Y(T(a)) > 0$ (the big image property);
- There are constants $C > 0$, $\theta \in (0, 1)$ such that

$$|\log \varphi(y_a) - \log \varphi(y'_a)| \leq C\, d_\theta(y_a, y'_a) \text{for all } y_a, y'_a \in a \text{ and for all } a \in \{a\}. \tag{10}$$

See, for instance, [1, Chapter 4] and [2] for background on Gibbs-Markov maps.

Given an observable $v : Y \to \mathbb{R}$, let

$$D_a v = \sup_{y, y' \in a,\, y \neq y'} |v(y) - v(y')|/d_\theta(y, y'), \qquad |v|_\theta = \sup_{a \in \{a\}} D_a v.$$

The space $\mathcal{B}_\theta \subset L^\infty(\mu_Y)$ consisting of the observables $v : Y \to \mathbb{R}$ such that $|v|_\theta < \infty$ with norm $\|v\|_{\mathcal{B}_\theta} = |v|_\infty + |v|_\theta < \infty$ is a Banach space. The transfer operator $R : L^1(\mu_Y) \to L^1(\mu_Y)$, defined by

$$\int_Y R^n v w \, d\mu_Y = \int_Y v w \circ T_Y^n \, d\mu_Y,\ n \geq 1, v \in L^1(\mu_Y), w \in L^\infty(\mu_Y)$$

has a spectral gap in the Banach space $\mathcal{B}_\theta \subset L^\infty(\mu_Y)$ (see, [1, Chapter 4]). In particular, this means that the spectral radius is 1 and 1 is also a simple eigenvalue, isolated in the spectrum of R.

A consequence of the spectral gap is that for $v \in \mathcal{B}_\theta$ (again, see, [1, Chapter 4] for details),

$$R^n v = \int_Y v d\mu_Y + Q^n v, \text{ where } \|Q^n\|_{\mathcal{B}_\theta} \leq \delta_0^n, \text{ for some } \delta_0 \in (0, 1). \tag{11}$$

It is known that this further allows one to obtain exponential decay of correlation for 'good' functions.

Using a perturbed version of R to obtain CLT and LLT for

$$g_n = \sum_{j=0}^{n-1} g \circ T_Y^j \text{ with } g : Y \to \mathbb{Z}^d, g|_a = C_a \text{ where } C_a \text{ is constant depending on } a,$$

$$\mu_Y(g = wN) = \frac{c_w}{N^3} + O\left(\frac{1}{N^4}\right) \text{ as } N \to \infty, \text{ for some } c_w > 0 \text{ and } w \in \mathbb{Z}^d. \tag{12}$$

Similar to the iid, we assume further that

$$\sum_{w \in \mathbb{Z}^d} c_w w \cdot w^T = A \text{ for some positive symmetric matrix } A. \tag{13}$$

So $A_{ij} = \sum_{w\in\mathbb{Z}^d} c_w w_i w_j$.

Much more relaxed assumptions on g are considered in [3], but here we are only interested in the 'displacement κ scenario'. The Fourier transform of $g_n = \sum_{j=0}^{n-1} g \circ T^j$ is

$$\mathbb{E}_{\mu_Y}(e^{it\cdot g_n}) = \int_Y e^{it\cdot g_n} 1\, d\mu_Y.$$

For $v \in \mathcal{B}_\theta$, define the **twisted transfer operator**

$$R_t v = R(e^{it\cdot g} v), \qquad t \in \mathbb{R}^d.$$

Note that $R_0 \equiv R$.

A calculation (see, for instance, [24, Lemma 3.2]) shows that $\int_Y e^{itg_n} v\, w \circ T^n d\mu_Y = \int_Y R_t^n v\, w\, d\mu_Y$. Taking $v = w = 1$, we get

$$\mathbb{E}_{\mu_Y}(e^{it\cdot g_n}) = \int_Y R_t^n 1 d\mu_Y.$$

This is the analogue of $\psi(t)^n$ in the i.i.d. set up. To obtain LLT one starts from (after using the Fourier inversion formula)

$$\mu_Y(g_n = N) = \frac{1}{(2\pi)^d} \int_{[-\pi,\pi]^d} e^{-it\cdot N} \left(\int_Y R_t^n 1 d\mu_Y \right) dt.$$

Recalling the 'message' in the i.i.d. set up, we need to

Understand the asymptotic behaviour of R_t^n as $t \to 0$

Unlike in the iid set up, for the proof of LLT, one needs to justify that the analysis can be reduced to a neighborhood of $0 \in \mathbb{R}^d$. (see text around equation (16) below).

Similar ingredients used to obtain (11) (see [2] and also the survey [24]) allows one to write

$$\begin{gathered}\text{There exists } \delta > 0 \text{ so that for all } t \in B_\delta(0) \text{ and for all } v \in \mathcal{B}_\theta, \\ R_t^n v = \lambda_t^n \Pi_t v + Q_t^n v, \text{ where } \|Q_t^n\|_{\mathcal{B}_\theta} \le \delta_0^n, \text{ for some } \delta_0 \in (0,1),\end{gathered} \tag{14}$$

where

$(\lambda_t)_{t\in B_{\delta_1}(0)}$ is a family of eigenvalues with $\lambda_0 = 1$

$$(\Pi_t)_{t\in B_{\delta_1}(0)} \text{ is a family of eigenprojections (operators on } \mathcal{B}_\theta \text{) with } \Pi_0 v = \int_Y v\, d\mu_Y. \tag{15}$$

For the LLT one also needs to justify that the analysis can be reduced to a neighborhood of $0 \in \mathbb{R}^d$. We will not go into this rather technical detail, but just mention that a consequence of the so-called **aperiodicity condition** is that, given δ as in (14),

$$\|R_t^n\|_{\mathcal{B}_\theta} \le \varepsilon_0^n \text{ for some } \varepsilon_0 \in (0,1) \text{ and for all } t \notin B_\delta(0). \tag{16}$$

For the details on the aperiodicity condition we refer to [2, Definition 3.1]. For the purpose of LLT we will simply assume that (16) holds.

Continuity of the family of operators $(R_t)_{t\in\mathbb{R}^d}$.

Given that $g \in L^1(\mu_Y)$ (a consequence of (12)) and that $\mathcal{B}_\theta \subset L^\infty(\mu_Y)$, there is an easy proof of a useful continuity estimate in t for R_t. For more details on continuity estimates for the family of operators $(R_t)_{t\in\mathbb{R}^d}$ we refer to [2, 3].

Lemma 3.3 *Assume* (12). *Then* $\|R_t - R\|_{\mathcal{B}_\theta} \le C|t|\,\|g\|_{L^1(\mu_Y)}$, *for some* $C > 0$.

Proof.(Sketch) Here we work with an equivalent formula for R, namely the *pointwise definition of the transfer operator*. Recall $\varphi = \frac{d\mu_Y}{d\mu_Y \circ T_Y} : Y \to \mathbb{R}$. Then we can write

$$Rv(y) = \sum_{x\in T_Y^{-1}y} \varphi(x)v(x) = \sum_{a\in\{a\}} \varphi(y_a)v(y_a),$$

where $y_a \in a \cap T_Y^{-1}y$. The first identity relies on a change of variables: see, for instance, the example in [24, Section 3].

Similarly, $R_t v(y) = \sum_{a\in\{a\}} \varphi(y_a)e^{it\cdot g(y_a)}v(y_a)$. One property of φ is that

$$\varphi(y_a) \le C\mu(a) \text{ for some } C > 0 \text{ and for all } y_a \in a \text{ and for all } a \in \{a\}. \tag{17}$$

Recalling that $g|_a = C_a$, where C_a is a constant depending on a,

$$\begin{aligned} |R_t - R|_\infty &\le |v|_\infty \sum_{a\in\{a\}} \varphi(y_a)\left|e^{it\cdot g(y_a)} - 1\right| \le |v|_\infty\,|t| \sum_{a\in\{a\}} \varphi(y_a)C_a \\ &\le |v|_\infty\,|t| \sum_{a\in\{a\}} \mu(a)C_a = |v|_\infty \|g\|_{L^1(\mu_Y)}\,|t|. \end{aligned}$$

where in the last inequality we have used (17).

Regarding the $|\,|_\theta$ seminorm, we note that for every $a \in \{a\}$, $|e^{it\cdot g}|_a|_\theta = 0$ since g is constant on partition elements. Hence,

$$\begin{aligned} |R_t - R|_\theta \le\ & |v|_\infty \sum_{a\in\{a\}, y_a, y'_a\in a} |\varphi(y_a) - \varphi(y'_a)|\left|e^{it\cdot g(y_a)} - 1\right| \\ &+ \sum_{a\in\{a\}, y_a, y'_a\in a} \varphi(y_a)|v(y_a) - v(y'_a)|\left|e^{it\cdot g(y_a)} - 1\right|. \end{aligned}$$

Using (10) and (17), we can bound the first term in the equation above by $|v|_\infty$ (up to a multiplicative constant independent of a). The second term is bounded (up to a multiplicative constant independent of a) by $|t|\,|v|_\theta$ due to definition of $|v|_\theta$ and (17). □

An immediate consequence of Lemma 3.3 is

$$|\lambda_t - \lambda_0| \le C_0|t|, \quad \|\Pi_t - \Pi\|_{\mathcal{B}_\theta} \le C_1|t|, \tag{18}$$

for constants $C_0, C_1 > 0$ and for all $t \in B_{\delta_1}(0)$.

Combining (15), the continuity estimate for Π_t in (18) and formula (14) for R_t^n, we arrive at

$$\begin{aligned} R_t^n v &= \lambda_t^n \Pi_t v + O(\delta_1^n) = \lambda_t^n \Pi_t v + O\left(|t|\,|\lambda_t^n|\right) + O(\delta_1^n) \\ &= \lambda_t^n \int_Y v\, d\mu_Y + O\left(|t|\,|\lambda_t^n|\right) + O(\delta_1^n). \end{aligned} \tag{19}$$

Recall that $\int_Y R_t^n 1\, d\mu_Y$ is the analogue of $\psi(t)^n$ in the i.i.d. set up. Following the message from the i.i.d. set up, that is that we want to prove that for t small,

$$\text{The asymptotic behavior of } \int_Y R_t^n 1\, d\mu_Y \text{ is of the form } \exp n\left(-\frac{\log(1/|t|)\, At\cdot t}{2} + O(t^2)\right),$$

for A a matrix as in (13).

A look at (19) tells us that if we can show that

$$\lambda_t^n \text{ is of the form } \exp n\left(-\frac{\log(1/|t|)\, At\cdot t}{2} + O(t^2)\right)$$

we are done. Since $\mathcal{B}_\theta \subset L^\infty(\mu_Y)$ in the situation of Gibbs-Markov maps, this is not hard.

The basic idea is that at the 'level of the eigenvalue', we can decompose λ_t into

- the Fourier transform of an i.i.d. process
- an error term that we can control.

Let $v_t = \frac{\Pi_t 1}{\int_Y \Pi_t 1\, d\mu_Y}$ be the normalized eigenvector, that is $\int_Y v_t\, d\mu_Y = 1$, associated with λ_t. Write $v_0 = 1$. Starting with $R_t v_t = \lambda_t v_t$ and integrating,

$$\begin{aligned} \lambda_t &= \int_Y R_t v_t\, d\mu_Y = \int_Y R_t 1\, d\mu_Y + \int_Y (R_t - R)(v_t - 1)\, d\mu_Y \\ &= \int_Y e^{it\cdot g} 1\, d\mu_Y + \int_Y (R_t - R)(v_t - 1)\, d\mu_Y. \end{aligned} \tag{20}$$

The first term in (20) : see the calculation leading to (7):

$$1 - \int_Y e^{it\cdot g} 1\, d\mu_Y = \log(1/|t|)\, At\cdot t + O(t^2)$$

with the matrix A is as in (13).

The remaining term $\int_Y (R_t - R)(v_t - 1)\, d\mu_Y$ is easy to control. By (18), $\|\Pi_t - \Pi\|_{\mathcal{B}_\theta} = O(|t|)$. This implies that $\|v_t - 1\|_{\mathcal{B}_\theta} = O(|t|)$. By Lemma 3.3, $\|R_t - R\|_{\mathcal{B}_\theta} = O(|t|)$. Crucially using that $\mathcal{B}_\theta \subset L^\infty(\mu_Y)$,

$$\left|\int_Y (R_t - R)(v_t - 1)\, d\mu_Y\right| = O(|t|^2).$$

As a consequence,

$$1 - \lambda_t = \log(1/|t|)\, At\cdot t + O(t^2) \text{ and thus, } \lambda_t^n = \exp n\left(-\frac{\log(1/|t|)\, At\cdot t}{2} + O(t^2)\right),$$

as desired. From here onward, the proof of non-standard CLT goes exactly as in the i.i.d. setup. The same applies to LLT under assumption (16).

3.2.2 Family of transfer operator on the Young tower of the Sinai billiard maps. Non-standard CLT and LLT

Recall from Subsection 3.1.2 that the Young tower for the Sinai billiard map can be reduced to **the one dimensional tower** $(\Delta, T_\Delta, \mu_\Delta)$. Recall that σ is the first return of the tower map T_Δ to the base Δ_0 and that $F = T_\Delta^\sigma$ is the first return map.

The base map (Δ_0, F, ν_0) is Gibbs-Markov. While one cannot obtain the desired limit laws solely by working with the base map F, one can take advantage of the Gibbs-Markov structure by

Relating quantities on the tower to quantities on the base.

In particular, one relates perturbed transfer operators along with eigenfamilies with the ones on the base.

The relevant work to the problem treated here is [6].

Transfer operator for the tower map $(\Delta, T_\Delta, \mu_\Delta)$

The main difference from the easier set of Gibbs-Markov maps is that the transfer operator $P : L^1(\mu_\Delta) \to L^1(\mu_\Delta)$, defined by

$$\int_Y P^n vw \, d\mu_\Delta = \int_Y vw \circ T_\Delta^n \, d\mu_\Delta, \; n \geq 1, v \in L^1(\mu_\Delta), w \in L^\infty(\mu_\Delta)$$

has a spectral gap in a Banach space $\mathcal{B}_\Delta \subset L^p(\mu_\Delta)$ for some $p > 1$. This was established in [45] based on the works [48, 10]. We do not provide the details on the Banach space here, but refer to [45], see also [6] or [41, Section 5.1].

Let $\|\cdot\|_{\mathcal{B}_\Delta}$ be the norm on $\mathcal{B}_\Delta$. The spectral gap of P on $\mathcal{B}_\Delta$, allows one to write

$$P^n v = \int_\Delta v d\mu_\Delta + Q^n v, \text{ where } \|Q^n\|_{\mathcal{B}_\Delta} \leq \delta_0^n, \text{ for some } \delta_0 \in (0,1). \tag{21}$$

Perturbed version of the transfer operator by the version of κ on Δ.

Let $\hat{\kappa} : \Delta \to \mathbb{Z}^d$ be the version of κ on Δ. Such a version of κ on Δ exists because κ is constant on stable curves. So, collapsing stable/unstable curves does not create a problem in reducing to the one dimensional tower (see, for instance, [41, Section 3]). For any $N \in \mathbb{Z}^d$,

$$\mu(\kappa_n = N) = \mu_\Delta(\hat{\kappa}_n = N), \quad \hat{\kappa}_n = \sum_{j=0}^{n-1} \hat{\kappa} \circ T_\Delta^j.$$

Define the perturbed operator $P_t v = P(e^{it\cdot\hat{\kappa}} v)$, $t \in \mathbb{R}^d$. As in the Gibbs-Markov case discussed previously, for the LLT, one needs to look at

$$\mu_\Delta(\hat{\kappa}_n = N) = \frac{1}{(2\pi)^d} \int_{[-\pi,\pi]^d} e^{-itN} \left(\int_\Delta P_t^n 1 d\mu_\Delta \right) dt. \tag{22}$$

By argument similar to those used for obtaining (21),

$$\begin{gathered}\text{There exists } \delta > 0 \text{ so that for all } t \in B_\delta(0) \text{ and for all } v \in \mathcal{B}_\Delta, \\ P_t^n v = \lambda_t^n \Pi_t v + Q_t^n v, \text{ where } \|Q_t^n\|_{\mathcal{B}_\theta} \leq \delta_0^n, \text{ for some } \delta_0 \in (0,1)\end{gathered} \tag{23}$$

where

$(\lambda_t)_{t\in B_{\delta_1}(0)}$ is a family of eigenvalues with $\lambda_0 = 1$

$(\Pi_t)_{t\in B_{\delta_1}(0)}$ is a family of eigenprojections (operators on $\mathcal{B}_\Delta$) with $\Pi_0 v = \int_Y v\, d\mu_Y$.

As established in [44] for the **finite horizon** case, $\hat\kappa$ is aperiodic. The same proof carries over to the infinite horizon case, as noted in [45]. Given δ as in (23),

$$\|P_t^n\|_{\mathcal{B}_\Delta} \le \varepsilon_0^n \text{ for some } \varepsilon_0 \in (0,1) \text{ and for all } t \notin B_\delta(0). \tag{24}$$

So far everything is similar to the easier set up of Gibbs-Markov maps, meaning that we can reduce the calculation to $t \in B_\delta(0)$. There is **one crucial difference:** $\mathcal{B}_\Delta \not\subset L^\infty(\mu_\Delta)$**.**

As in the Gibbs-Markov set up, we write

$$1-\lambda_t = \int_Y (1-e^{it\cdot\hat\kappa})1\, d\mu_\Delta - \int_\Delta (P_t - P)(v_t - 1)\, d\mu_\Delta. \tag{25}$$

As before, plays the role of the Fourier transform of an i.i.d. process and it can be estimated as in the i.i.d. set up. As established in [45, Proposition 6], there exists a finite set $\mathcal{S} = \{(L,w) \in (\mathbb{Z}^d, \mathbb{Z}^d) : \gcd(w_i, w_j) = 1 \text{ for all } 1 \le i < j \le d\}$ which parametrizes the set of corridors so that

$$\mu(\hat\kappa = L + Nw) = c_{L,w} N^{-3}(1+o(1)), \text{ with } c_{L,w} > 0. \tag{26}$$

It can be extracted from [45] that

$$\begin{aligned}\int_Y (1-e^{it\cdot\hat\kappa})1\, d\mu_\Delta &= \log(1/|t|)\left(\Big(\sum_{(L,w)\in\mathcal{S}} c_{L,w} w\cdot w^T\Big)t\right) t\,(1+o(1))\\ &= \log(1/|t|)\, \Sigma t\cdot t\,(1+o(1)),\end{aligned}$$

for a $d\times d$ matrix Σ with $\det\Sigma \neq 0$. If $d = 1$, then Σ is a scalar.

Showing that $\left|\int_\Delta (P_t - P)(v_t - 1)\, d\mu_\Delta\right|$ **is** $o(t^2\log(1/|t|))$ is seriously more complicated than in the Gibbs-Markov case. It is not hard to control $\|v_t - 1\|_{L^q}$ for some $q < p$, but this fact cannot be used directly. It helps to develop an intuition and it can be used in several much more complicated estimates.

The fact that $\left|\int_\Delta (P_t - P)(v_t - 1)\, d\mu_\Delta\right| = o(t^2\log(1/|t|))$ was established in [45] using:

- an abstract result on the tower Δ, namely [6, Theorem 3.5].

We do not recall the details of [6, Theorem 3.5], but mention that the verification of the conditions of the abstract result [6, Theorem 3.5] is not at all trivial. In the reminder of the section we review briefly the estimate of λ_t with an error term, which gives LLT and mixing with rates.

The verification of the conditions of the abstract result [6, Theorem 3.5] relies on the of the so called 'double probability', among other ingredients. Namely, as established in [45, Propositions 11–12] and [45, Lemma 16], there exists $\varepsilon > 0$ such that, for every V,

$$\mu_\Delta(A_{n,V}) = O(n^{-3-\varepsilon}) \quad \text{with } A_{n,V} := \left\{\hat\kappa = n,\ \exists |j| \le V\log(n+2),\ |\hat\kappa\circ f^j| > n^{4/5}\right\} \tag{27}$$

In words, the joint probability of the event $\hat\kappa = n$ and the event that $\hat\kappa$ is almost equally large within approximately $\log n$ iterates (that is, $|j| \le V\log(n+2)$, $|\hat\kappa\circ f^j| > n^{4/5}$) is smaller than the probability of the single event $\hat\kappa = n$. This is sufficient to gain a factor $n^{-\varepsilon}$ for some fixed $\varepsilon > 0$.

Putting together the estimates for $\int_Y (1-e^{it\cdot\hat\kappa})1\, d\mu_\Delta$ and $\left|\int_\Delta (P_t - P)(v_t - 1)\, d\mu_\Delta\right|$ and recalling (25), one obtains $1-\lambda_t = \log(1/|t|)\, \Sigma t\cdot t\,(1+o(1))$, which is all that is needed to obtain LLT.

Error rates in LLT as in [41]. In the reminder of this section, we provide the details for the use of (27) for obtaining error rates in the LLT as [41]. Recalling the i.i.d. set up and the Gibbs-Markov scenario, we need to establish that

$$1 - \lambda_t = \log(1/|t|)\, \Sigma t \cdot t \,+ O(t^2).$$

Recalling (25) we need to show that

(a) $\int_Y (1 - e^{it\cdot\hat{\kappa}})1\, d\mu_\Delta = \log(1/|t|)\, \Sigma t \cdot t \,+ O(t^2)$;

(b) $\left|\int_\Delta (P_t - P)(v_t - 1)\, d\mu_\Delta\right| = \left|\int_\Delta (e^{it\hat{\kappa}} - 1)(v_t - 1)\, d\mu_\Delta\right| = O(t^2)$.

Showing (a). This part is the easy part. With the same notation, [41, Lemma 4.2] shows that with the same notation as in (26),

$$\mu(\hat{\kappa} = L + Nw) = c_{L,w} N^{-3} + O(N^{-4}), \text{ with } c_{L,w} > 0.$$

Given this expnasion of the tail, item (a) follows essentially as in the computation in the i.i.d. set up treated above.

Showing (b). This is not at all trivial. Improving by just $O(t^2)$ turns out to be quite challenging. Recall that $v_t = \frac{\Pi_t 1}{\int_\Delta \Pi_t 1\, d\mu_\Delta}$. Write

$$V(t) = \int_\Delta (e^{it\hat{\kappa}} - 1)(v_t - 1)\, d\mu_\Delta.$$

It is easy to see that $\mu_\Delta((R_t - R_0)(1_\Delta)) = O(t)$. However, showing that the same holds for Π_t is an entirely different story. Building on [6, Section 3], [41, Proposition 5.3] implies that $\mu_\Delta((\Pi_t - \Pi_0)(1_\Delta)) = O(t)$. In particular, to obtain this result, the authors 'made sense' of a derivative at 0 for Π_t. (In order to obtain error terms in LLT and mixing, [41, Proposition 5.3] obtains higher order terms in $\|\Pi_t - \Pi_0\|_{L^q} = O(|t|)$ for a range of $q \in [1, 2)$.)

As mentioned already, an important part of the strategy in [6] is to relate quantities on the tower with quantities on the base. We start from the same expression Π_t.

Let $\pi_0 : \Delta \to \Delta_0$ be the vertical projection from Δ to the base Δ_0 defined by $\pi_0(x, \ell) = (x, 0)$. Let $\omega : \Delta \to \mathbb{N}$ be the level map defined by $\omega(x, \ell) = \ell$. Thus $x = T_\Delta^{\omega(x)}(\pi_0(x))$ for all $x \in \Delta$.

$$\Pi_t v(x) = \lambda_t^{-\omega(x)} P_t^{\omega(x)}(\Pi_t v)(x) = \lambda_t^{-\omega(x)} e^{it\cdot\hat{\kappa}_{\omega(x)}(\pi_0(x))} \Pi_t(v) \circ \pi_0(x)\,. \tag{28}$$

The first equality in (28) is a consequence of $P_t^{\omega(x)} \Pi_t v(x) = \lambda_t^{\omega(x)} \Pi_t v(x)$.

Using that $\mu_\Delta((\Pi_t - \Pi_0)(1_\Delta)) = O(t)$, one obtains that

$$V(t) = \int_\Delta (1 - e^{it\cdot\hat{\kappa}})(\Pi_t - \Pi_0)(1_\Delta)\, d\mu_\Delta + O(t^2). \tag{29}$$

Below, we summarize the steps of [41, Proof of Lemma 6.1] in order to highlight the use of the 'double probability' (27). The formula (28) is heavily used.

From (29), and (28) we get

$$\left|\int_\Delta (1 - e^{it\cdot\hat{\kappa}})(\Pi_t - \Pi_0)(1_\Delta)\, d\mu_\Delta\right| \le J_1(t) + J_2(t) + J_3(t)\,, \tag{30}$$

where

$$J_1(t) := \left|\int_\Delta (1-e^{it\cdot\hat\kappa(x)})(e^{it\cdot\hat\kappa_{\omega(x)}(\pi_0(x))}-1)\,d\mu_\Delta(x)\right|,$$
$$J_2(t) := \int_\Delta \left|(1-e^{it\cdot\hat\kappa(x)})e^{it\hat\kappa_{\omega(x)}}(\Pi_t-\Pi_0)(1_\Delta)\right|\circ\pi_0(x)\,d\mu_\Delta(x),$$
$$J_3(t) := \int_\Delta \left|(1-e^{it\cdot\hat\kappa(x)})(\lambda_t^{-\omega(x)}-1)e^{it\kappa_{\omega(x)}}\Pi_t(1_\Delta)\right|\circ\pi_0(x)\,d\mu_\Delta(x).$$

The terms J_2, J_3 are shown to be $O(t^2)$, see [41, Proof of Lemma 6.1] (which are building blocks in obtaining the previously mentioned [41, Proposition 5.3]). This is based on several additional estimates, which requires an understanding of the derivative of $\Pi_t(v)\circ\pi_0(x)$ in t, evaluated at 0.

Here we just look at J_1, which can be handled via the 'double probability' (27), and also exponential tail of $\mu_\Delta(\sigma > m)$. Note that

$$J_1(t) = \left|\int_{\Delta_0}\sum_{k=0}^{\sigma-1}(1-e^{it\cdot\hat\kappa\circ T_\Delta^k})(1-e^{it\cdot\hat\kappa_k})\,d\mu_\Delta\right| \le t^2\int_{\Delta_0}\sum_{k=1}^{\sigma-1}|\hat\kappa\circ T_\Delta^k|\,|\hat\kappa_k|\,d\mu_\Delta$$

Thus, it remains to show that

$$\int_{\Delta_0}\sum_{k=1}^{\sigma-1}|\hat\kappa\circ T_\Delta^k|\,|\hat\kappa_k|\,d\mu_\Delta \text{ is bounded.} \tag{31}$$

Proof. (of equation (31))

For any $y\in Y$, we write $\ell(y)$ for the largest integer in $\{1,...,\sigma(y)-1\}$ such that

$$N(y) := \sup_{k=1,\dots,\sigma(y)-1}|\hat\kappa\circ T_\Delta^k(y)| = |\hat\kappa\circ T_\Delta^{\ell(y)}(y)|.$$

Set

$$Y_n := \Delta_0\cap\{\hat\kappa\circ T_\Delta^{\ell(y)} = n\},$$
$$Y_n' := \{y\in Y_n \;:\; \sigma(y) < b\log(n+2)\},$$

and

$$Y_n^{(0)} := \left\{y\in Y_n' \;:\; \forall j<\sigma(y),\ |\hat\kappa\circ f^j|\le n^{4/5}\right\}.$$

Notice that

$$\begin{aligned}
\int_Y\sum_{k=1}^{\sigma-1}|\hat\kappa\circ f^k|\,|\hat\kappa_k|\,d\mu_\Delta &\le \sum_{n\ge0}\int_{Y_n'}\sum_{k=1}^{b\log(n+2)-1}|\hat\kappa\circ f^k|\,|\hat\kappa_k|\,d\mu_\Delta + \sum_{n\ge0}\int_{Y_n\setminus Y_n'}n^2\sigma\,d\mu_\Delta \\
&\le \sum_{n\ge0}\int_{Y_n^{(0)}} b\,n^{9/5}\log(n+2) + \sum_{n\ge0}\int_{Y_n'\setminus Y_n^{(0)}} n^2 b\log(n+2)\,d\mu_\Delta + \sum_{n\ge0}n^2\mathbb{E}_{\mu_\Delta}[\sigma 1_{Y\setminus Y_n'}] \\
&\le \sum_{n\ge0} b\,n^{9/5}\log(n+2)\mu_\Delta(Y_n') + \sum_{n\ge0}n^2 b\log(n+2)\mu_\Delta(Y_n'\setminus Y_n^{(0)}) \\
&\qquad + \sum_{n\ge0}n^2\sum_{m\ge b\log(n+2)}\mu_\Delta(\sigma>m).
\end{aligned}$$

It is known that $\mu_\Delta(\sigma > m) \le C\theta_1^m$ for $\theta_1 \in (0,1)$. The last term of this displayed equation is less than

$$\sum_{n\ge 0} n^2 \sum_{m\ge b\log(n+2)} C_1\theta_1^m \le \sum_{n\ge 0} O(n^2\theta_1^{b\log(n+2)}) < \infty$$

for b large enough. By the asymptotic of (26), the other terms are dominated by

$$\sum_{n\ge 0} O\left(n^{9/5}\log(n+2) \sum_{m=0}^{b\log(n+2)} \mu_\Delta(\hat\kappa = m)\right) + \sum_{n\ge 0} n^2\log(n+2) \sum_{m=0}^{b\log(n+2)} \mu_\Delta(A_{n,b})$$
$$\le \sum_{n\ge 0} O\left(n^{9/5-3}(\log(n+2))^2\right) + \sum_{n\ge 0} n^{2-3-\varepsilon}(\log(n+2))^2 < \infty,$$

where $A_{n,b}$ is from (27). □

Mixing and mixing LLT for 'suitable' class of functions The method of perturbed transfer operators (as before this is the Nagaev-Guivarch-Aaronson-Denker' method), allows one to work with a 'suitable' class of observables.

Instead of working with $v = w = 1$ as in (22) we can write for $v \in L^1(\mu_\Delta)$ and $w \in L^\infty(\mu_\Delta)$:

$$\int_\Delta v 1_{\hat\kappa_n = N} w \circ T_\Delta^n \, d\mu_\Delta = \frac{1}{(2\pi)^d} \int_{[-\pi,\pi]^d} e^{-itM} \left(\int_\Delta P_t^n v w \circ T_\Delta^n \, d\mu_\Delta \right) dt. \tag{32}$$

Provided that $v \in \mathcal{B}_\Delta$ one would proceed as before by exploiting (23) and the approach previously described.

A statement of the form (32) is called a Mixing Local Limit Theorem (MLLT). But the meaningful question is

What are suitable observables v, w in the setting of LG?

This question was answered for the finite and also infinite horizon by Pène in [38], see also [40, Section 3] for a summary of the main results and additional challenges. For a suitable class of observables, when treating error terms in MLLT we refer to [41].

Recalling what we summarized in Subsection 2, MLLT translate or lead (again, depending on the class of observables) to Mixing for the LG.

4 Mixing and Mixing Local Limit Theorem (MLLT) for the periodic LG flows (continuous time)

4.1 Describing the continuous time LG

Recall Figures 1 and 3 for the tubular LG finite or infinite horizon and also Figure 4 for the two-dimensional LG.

The dynamics is the same but this time we look at it in continuous time. As such, the quantity that we previously ignored, namely the **flight time** $\tau : M \to \mathbb{R}_+$ which gives the time between consecutive collisions, will play an essential role.

Consider Figure 11 describing LG flow via a suspension flow. In the flow case, instead of looking at the extension of the discrete dynamics, the discrete time Sinai billiard $T : M \to M$, we look at a suspension over it. We will refer to this object as the **Sinai billiard flow** and we will denote this by ϕ_t. The Poincaré map of ϕ_t is the Sinai billiard map (T, M, μ) described in the previous sections. Recall from Subsection 3.2.2 that points $x \in M$ are $x = (q, v)$, where q is the position (on the boundary of scatterers) and v is the unit velocity vector parametrized by the angle $\varphi \in [-\pi/2, \pi/2]$.

Recalling the meaning of the flight time τ,

$$T = \phi_\tau \text{ and } \widehat{M} = \{(x, u) : 0 \le u \le \tau(x)\}/\sim \text{ with } (x, \tau(x)) \sim (T(x), 0).$$

The flow ϕ_t preserves the probability measure $\nu = (\mu \times Leb)/\bar{\tau}$ where $\bar{\tau} = \int_M \tau \, d\mu$.

In this notation, we have the following identification

$$T(x) = (\phi_\tau)(x) = \phi_{\tau(x)}(x) \,.$$

We set $\tau_n := \sum_{j=0}^{n-1} \tau \circ T^j$, with the usual convention $\tau_0 := 0$.

For any $x = (q, v) \in M$, we set $N_t(x) \in \mathbb{N}_0$ for the *collisions number* in the time interval $(0, t]$ starting from x. For $x \in M$, this quantity satisfies

$$\tau_{N_t(x)}(x) \le t < \tau_{N_t(x)+1}(x) \,.$$

Furthermore, $N_t(\phi_u(x)) = N_{t+u}(x)$ for all $x \in M$ and all $u \in [0, \tau(x))$ and any $t \in [0, +\infty)$. With these notations, the Sinai billiard flow $(\widehat{M}, (\phi_t)_t, \nu)$ can be represented as

$$\phi_t : \widehat{M} \to \widehat{M}, \quad \phi_t(x, u) = (T^{N_{t+u}(x)}(x), t + u - \tau_{N_{t+u}(x)}(x)).$$

LG flow as a $\mathbb{Z}^d$-extension by $\kappa : M \to \mathbb{Z}^d$ The blue arrows in Figure 11 show the transition from cell to cell, where the dynamics on the cell is given by the Sinai billiard flow $\phi_t : \widehat{M} \to \widehat{M}$.

A useful representation of the LG flow is one that similar to the LG map. This time, we write $\widetilde{M} = \widehat{M} \times \mathbb{Z}^d$ and define

$$\Phi_t(x, u, \ell) = (T^N x, \ell + \kappa_N(x), u + t - \tau_N(x)) \quad \text{for } N = N_t(x, u) \text{ with } \tau_N(x) \le t < \tau_{N+1}(x).$$

The flow Φ_t preserves the infinite measure $\tilde{\nu} = \nu \times Leb_{\mathbb{Z}^d}$.

4.2 Mixing for the Lorentz gas flow Φ_t seen as a suspension flow via MLLT for κ

The meaning of finite horizon for the map/discrete time was clarified in Subsection 3.2.2. We said that this means that κ is bounded. In the finite horizon case, the flight time is also bounded.

The meaning of infinite horizon for the map/discrete time was also clarified in Subsection 3.2.2 and as mentioned there $\kappa \notin L^2(\mu)$. In the flow scenario, τ has the same tail as κ.

In Section 2, we mentioned that all limit laws obtained for κ translate into limit laws for the flight function $V : M \to \mathbb{R}^d$ that gives the distance in $\mathbb{R}^d$ between collisions. Recall that

$$V(x) = \kappa(x) + H(x) - H(Tx),$$

where $H - H \circ T$ is referred to as a bounded (mean zero) coboundary. Clearly, $\mu(|\kappa| > N)$ and $\mu(|V| > N)$ have the same asymptotics.

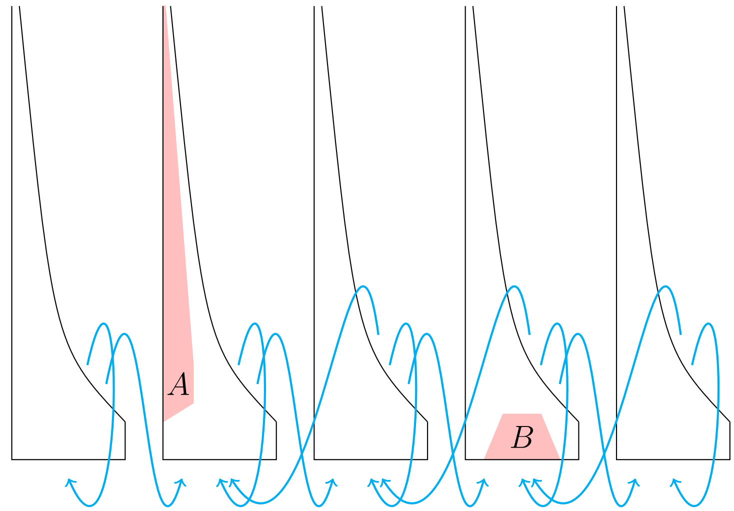


Figure 11: Lorentz gas flow via suspension flow.

For the planar Lorentz gas ($d = 2$), we can write

$$|V| = \tau, \text{ and thus } \mu(\tau > N) = \mu(|V| > N).$$

For the Lorentz tube ($d = 1$), the flight time τ can be very different from the modulus $\|\kappa\|$ of the displacement function. Indeed, a flow-line can wrap many times around the circle direction of $\mathbb{R} \times \mathbb{S}^1$ (making τ large), while staying in the same cell (so $\kappa = 0$). Nonetheless, the same type of tail for τ holds. We refer to [42].

Set

$$a_n = \begin{cases} n^{d/2}, & \text{in the finite horizon case,} \\ (n \log n)^{d/2} & \text{in the infinite horizon case,} \end{cases} \tag{33}$$

and recall that

$$\frac{\kappa_n}{a_n} \to^d \mathcal{N}(0, \Sigma) \tag{34}$$

To state MLLT for both finite and infinite horizons, we consider the following class of measurable sets.

Definition 4.1 *Let $\mathcal{F}$ be the class of measurable subsets A of $\widehat{M}$ of the form $A = \phi_I(A_0) = \{\phi_u(x),\, x \in A_0,\, u \in I\}$ that are represented in $\widehat{M}$ by $A_0 \times I \subset \widehat{M}$ (implying that $I \subset [0, \inf_{A_0} \tau)$), with $A_0 \subset M$ a measurable set satisfying $\mu(\partial A_0) = 0$ and with I a bounded interval.*

Theorem 4.2 *Let $A, B \in \mathcal{F}$ with $\mathcal{F}$ as in Def. 4.1 . Let a_t as defined in (33). Let K be a bounded subset of $\mathbb{R}^d$, let $w \in \mathbb{R}^d$ and $w_t \in \mathbb{R}^d$ be such that $\lim_{t \to +\infty} w_t / a_t = w$. Then*

$$a_t^d \; \nu\left(A \cap \{\phi_t \in B,\, \kappa_{N_t} \in w_t + K\}\right) \sim \Psi\,(w)\,\nu(A)\nu(B)\#((K + w_t) \cap \mathbb{Z}^d)\,,$$

where Ψ is the density of the Gaussian limit in (34).

In both the finite and infinite horizon case, the results are recent. In the finite horizon case, Theorem 4.2 was proved by Dolgopyat and Nándori [19] and complete expansions (error rates up to any order) were obtained by Dolgopyat, Nańdori and Pène in [20]. In the infinite horizon, Theorem 4.2 was proved by Pène and Terhesiu in [42].

Theorem 4.2 implies mixing the Lorentz gas flow $\widetilde{\phi}_t$
for compactly supported observables in the 'extended suspension'

In the **infinite horizon**, a main challenge was to consider observables NOT compactly supported observables in the 'extended suspension'. The essential feature of this very natural class of observables is that

The support of these observables may contain points with infinite flights.

There is no analogue of this in the finite horizon case.

We recall the statement of [42, Theorem 1.1] as to get the idea across.

Write $\mathcal{D}_2 := \mathbb{R}^2$ (for $d = 2$) or on the tube $\mathcal{D}_1 := \mathbb{R} \times \mathbb{S}^1$ (for $d = 1$), and let Ω_d be the set of positions in $\mathcal{D}_d$ that are not inside an obstacle. In the statement below, we look at the Lorentz gas flow $(\Phi_t)_t$ not via its representation as $\mathbb{Z}^d$ extension of the suspension flow, but directly on the billiard manifold. The flow Φ_t maps a point $(q, v) \in \widetilde{M}$ (corresponding to a couple position and velocity at time 0) to a point $\Phi_t(q, v) = (q_t, v_t) \in \widetilde{M}$ corresponding to the couple position and velocity at time t.

Theorem 4.3 *[42, Theorem 1.1] For all continuous compactly supported observables $f, g : \Omega_d \times \mathbb{S}^1 \to \mathbb{R}$,*

$$\int_{\widetilde{M}} f.g \circ \Phi_t \, d\widetilde{\nu} \sim \frac{\int_{\widetilde{M}} f \, d\widetilde{\nu} \int_{\widetilde{M}} g \, d\widetilde{\nu}}{(2\pi t \log t \det(\Sigma))^{\frac{d}{2}}}\,, \quad \text{as } t \to +\infty\,.$$

If f is a compactly supported observable on the billiard phase space $\Omega_d \times \mathbb{S}^1$, then its lift $f \circ \hat{\pi}^{-1}$ to a suspension flow need not be compactly supported. Here $\hat{\pi}(x, u) = \phi_u(x)$ is the projection from the suspension flow to the billiard flow space.

There are two mechanisms that enable the non-compactness of $\hat{\pi}^{-1}(\mathrm{supp}(f))$.

- The return to the Poincaré section is not a first but rather a "good" return. Then a flow-line can intersect $\mathrm{supp}(f)$ any number of times, say R, before a "good" return happens. Thus, $\hat{\pi}^{-1}(\mathrm{supp}(f))$ might contain (as many) R uniformly separated components.
- Even if the billiard manifold is compact, it can still allow infinite horizon. For example the Sinai billiard on a torus with a single round scatterer has this property.

  A flow-line can wrap any number of times (say R) around the torus (and intersecting $\mathrm{supp}(f)$) before hitting the scatterer. In this case, $\hat{\pi}^{-1}(\mathrm{supp}(f))$ might contain R uniformly separated components, too.

For an impression of the of observables considered in [42, Theorem 1.1], consider Figure 11: a function supported on B is compactly supported, while a function supported on A is not compactly supported in the extended suspension.

We state a light version of the mixing result along with a short proof.

Corollary 4.4 *(of Theorem 4.2) Let $n \in \mathbb{N}$ and set*

$$E_{\pm n} := \left\{ \Phi_{\pm u}(q+\ell,\vec{v}) \in \widetilde{M} \,:\, q \in \bigcup_{i=1}^{I} \partial O_i,\ u \in [0,n],\ \ell \in \mathbb{Z}^d,\ |\ell| \le |n| \right\}.$$

Then, for all observables $f, g : \widetilde{M} \to \mathbb{R}$ that are continuous μ-a.e. and supported respectively in E_{-n} and in E_n, we have

$$(a_t)^d \int_{\widetilde{M}} f.g \circ \Phi_t \, d\widetilde{\nu} \sim \int_{\widetilde{\mathcal{M}}} f \, d\widetilde{\nu} \int_{\widetilde{\mathcal{M}}} g \, d\widetilde{\nu} \, \Psi(0), \tag{35}$$

as $t \to +\infty$.

Proof. Let $A_0 \times I, B_0 \times I$ be two sets belonging to $\mathcal{F}$ defined in Defintion 4.1.

Let A, B be two sets of the form $A_0 \times I \times \{\ell_0\}$ and $B_0 \times J \times \{\ell_0'\}$ in $\widehat{\mathcal{M}} \times \mathbb{Z}^d$. We observe that

$$\widetilde{\nu}\left(A \cap \Phi_{-t}(B)\right) = \nu(\phi_I(A_0) \cap \{\phi_t \in \phi_J(B_0),\, \kappa_{N_t} = \ell_0' - \ell_0\}).$$

Consider the set up of Theorem 4.2 with

$$w_t = \ell \in \mathbb{Z}^d, \qquad w = 0, \qquad K = \{0\}.$$

Then

$$\forall \ell \in \mathbb{Z}^d, \quad (a_t)^d \nu\left(A \cap \{\phi_t \in B,\, \kappa_{N_t} = \ell\}\right) \sim \Psi\left(0\right) \nu(A) \nu(B), \tag{36}$$

Thus, it follows from (36) that

$$(a_t)^d \widetilde{\nu}\left(A \cap \Phi_{-t}(B)\right) \sim \widetilde{g}_d(0) \nu(\phi_I(A_0)) \nu(\phi_J(A_0)) = \Psi(0) \widetilde{\nu}(A) \widetilde{\nu}(B).$$

This result extends directly to any finite union of sets A, B as defined in this proof.

Note that the flow Φ_t is invertible. Thus, if f is supported in E_{-n}, then $f \circ \Phi_{-n}$ is supported on E_n. Therefore,

$$\int_{\widetilde{M}} f.g \circ \Phi_t \, d\nu = \int_{\widetilde{M}} f \circ \Phi_{-n}.g \circ \Phi_{t-n} \, d\nu,$$

since $a_{(t-n)}^d \sim a_t^d$. □

4.2.1 Main steps of the proof of Theorem 4.2 in the infinite horizon case

To get the idea, it suffices to understand how the proof goes assuming $K = \{0\}$ and assuming $w_t \in \mathbb{Z}^d$. The general case follows as in [42, Section 4.1].

Step 1 Recall from Definition 4.1 that A is a set of the form $A_0 \times I \subset \widehat{M}$ with $I \subset [0, \inf_{A_0} \tau))$ and $A_0 \subset M$. The same holds for B with J instead of I.

Write $y = (x, u) \in \widehat{M}$ and note that

$$\begin{aligned} &\nu\left(y \in A \cap \{\phi_t(y) \in B,\, \kappa_{N_t(y)}(x) = w_t\}\right) \\ &= \nu(x \in A_0, T^{N_t(y)}(x) \in B_0, u \in I, u + t - \tau_{N_t(y)}(x) \in J, \kappa_{N_t(y)}(x) = w_t) \\ &= \sum_{n=1}^{\infty} \mu(x \in A_0, T^n(x) \in B_0, u \in I, \tau_n(x) \in u + t - J, \kappa_n(x) = w_t) \\ &= \frac{1}{\mu(\tau)} \sum_{n \ge 0} \int_I Q_n(t,u) \, du, \end{aligned}$$

where

$$Q_n(t,u) := \mu\left(A_0 \cap T^{-n}B_0 \cap \{\kappa_n = w_t,\, \widetilde{\tau}_n \in u + t - n\mu(\tau) - J\}\right).$$

The message of this equation is: contrary to the discrete time case, τ_n is now going to play a crucial role. Instead of the single observable κ_n, one looks at the joint observable

$$\widehat{\Psi}_n = (\kappa_n, \widetilde{\tau}_n) \text{ where } \tilde{\tau} = \tau - \int_M \tau\, d\mu.$$

So, we can write

$$Q_n(t,u) = \mu\left(A_0 \cap T^{-n}(B_0) \cap \left\{\widehat{\Psi}_n \in (w_t, t - n\mu(\tau)) + \{0\} \times J_u\right\}\right),$$

with $J_u = u - J$.

Step 2 For L large, we split the sum as

$$\nu\left(A \cap \{\phi_t \in B,\, \kappa_{N_t} = w_t\}\right) = S_1(t,L) + S_2(t,L),$$

where

$$S_1(t,L) := \frac{1}{\mu(\tau)} \sum_{n\,:\,|n - t/\mu(\tau)| \le La_t} \int_I Q_n(t,u)\, du,$$

$$S_2(t,L) := \frac{1}{\mu(\tau)} \sum_{n\,:\,|n - t/\mu(\tau)| > La_t} \int_I Q_n(t,u)\, du.$$

The reason for this splitting is that S_1 will give the precise asymptotics via the LLT for $\widehat{\Psi}_n$, while S_2 can be shown to be negligible. More precisely, we obtained

Lemma 4.5 *(a)* $\lim_{L\to\infty} \lim_{t\to\infty} a_t^d S_1(t,L) = \Psi(0)\nu(A)\nu(B)$,
(b) $\lim_{L\to\infty} \limsup_{t\to\infty} a_t^d S_2(t,L) = 0$.

Step 3 Item (a) of Lemma 4.5 is a joint LLT, namely

Lemma 4.6 (Joint CLT for the billiard map) $a_n^{-1}\widehat{\Psi}_n \implies \mathcal{N}(0, \Sigma_{d+1})$ *as* $n \to \infty$*, where* Σ *is a real positive-definite* $(d+1) \times (d+1)$ *matrix.*

Step 4 Item (b) of Lemma 4.5 uses among other ingredients, a joint Local Large Deviation for $\widehat{\Psi}$.

Lemma 4.7 (Joint LLD for the billiard map) *Let* $U \subset \mathbb{R}^{d+1}$ *be an open ball. Then*

$$\mu(\widehat{\Psi}_n \in z + U) \ll \frac{n}{a_n^{d+1}} \frac{\log(2 + |z|)}{1 + |z|^2}$$

uniformly in $n \ge 1$ *and* $z \in \mathbb{R}^{d+1}$*.*

Lemma 4.7 is a generalization of LLD for κ obtained by Melbourne, Pène and Terhesiu in [37], which says that

$$\mu(\kappa_n = N) \ll C \frac{n}{a_n^d} \frac{\log(|N|)}{1 + |N|^2}$$

uniformly in $n \ge 1$ and $N \in \mathbb{Z}^d$. What matters here is the range: it holds for $|N| > a_n$. For $N \ge a_n$ this is a consequence for the LLT for κ (under the map T). What matters here is the range.

5 Other approaches beyond Young towers

The Young–tower framework has become a standard tool for proving limit theorems in Sinai billiards and Lorentz gases, but there are complementary approaches that are powerful in their own right. We briefly highlight three directions.

Functional–analytic (spectral) methods on anisotropic spaces. A complementary route is to study the Perron–Frobenius/transfer operator on carefully chosen *anisotropic Banach spaces*, proving a spectral gap and then deriving statistical properties from spectral perturbation theory. For planar Lorentz gases this has been developed in a series of works by Demers–Zhang, yielding spectral decompositions, decay of correlations and linear–response–type stability results [16, 17, 18] building on previous work of Demers and Liverani [14]. Robustness under perturbations for piecewise hyperbolic maps—key to billiards with singularities—was established in [14]. More recently, Demers and Liverani [15] introduces *projective cone* techniques that streamline Lasota–Yorke–type estimates for generalized dispersing billiards, giving a flexible path to spectral bounds beyond classical settings.

Kinetic limits and the Boltzmann–Grad program. Another viewpoint replaces long–time statistical questions by *low–density* scaling limits, linking the Lorentz gas to kinetic equations. For the periodic Lorentz gas, the rigorous Boltzmann–Grad limit was obtained in [32], with refined asymptotics in [33]. This program clarifies how free–path statistics and lattice geometry feed into kinetic transport. Within this circle of ideas, [34] analyzes superdiffusive behavior in periodic geometries, highlighting mechanisms tied to long corridors.

Random Lorentz gases and invariance principles beyond Boltzmann–Grad. Beyond periodic configurations, one can randomize the scatterers and ask for invariance principles under different scalings. The work [30] develops an invariance principle for the random Lorentz gas in regimes that go *beyond* the classical Boltzmann–Grad scaling, connecting microscopic geometry to macroscopic Brownian (or superdiffusive) limits.

In short. Spectral/functional–analytic techniques ([14, 16, 17, 18, 15]) provide a direct control of transfer operators and stability; kinetic–limit methods ([32, 33]) link the Lorentz gas to the Boltzmann–Grad program; and random–media results ([30]) extend invariance principles to nonperiodic settings.

Aperiodic geometries and ergodicity. Non periodic setting are difficult to study. Beyond periodic arrays, Lenci established ergodicity for *aperiodic* Lorentz gases: in two dimensions with finite horizon and mild nondegeneracy, *recurrence implies ergodicity*, and explicit recurrent aperiodic examples are constructed [27]. For infinite-horizon geometries and Lorentz tubes, Lenci and Troubetzkoy built aperiodic classes that are recurrent, uniformly hyperbolic, and ergodic (with K-mixing first-return maps). [28].

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